To: Families and Caregivers of CMSD Students:

CMSD continues to send regular updates on the services and supports we are providing during the unprecedented closure of schools, as part of a state-wide effort to contain the spread of the COVID-19 virus.

In addition to the grab-and-go meals we are providing at 22 school sites each day, CMSD is also distributing learning packets, and I want to personally emphasize the value of these academic enrichment materials that are handed out with meals and posted on the CMSD website: ClevelandMetroSchools.org.

Research shows that children learn best when learning is continuous, which is why CMSD educators are working hard to produce interesting and thought-provoking materials that will keep students engaged and that will keep their minds active during this long break from school.

Recognizing that students are used to a consistent school schedule, I strongly encourage you to work with your child to develop a routine at home, to make time and space for quiet reading and active engagement with their learning materials and to praise them for their attention to their studies and their personal growth.

CMSD’s Academic Enrichment Plan, posted on CMSD’s website, includes lessons and a recommended daily schedule for students at every grade level, from PreK to 12. Digital lessons can be accessed online and print materials are available for pickup at all meal sites.

Thank you for the opportunity to emphasize the importance of academic enrichment in our students’ experience during this unprecedented time away from school. And thank you for the important role you play every day in our shared commitment to the safety, growth and future of Cleveland’s children.

Thank you.

Eric S. Gordon  
CEO
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<thead>
<tr>
<th></th>
<th>April 6</th>
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<th>April 8</th>
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<td>Algebra I</td>
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<td>GOOD FRIDAY – NO ENRICHMENT ACTIVITIES TODAY</td>
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<td><strong>(40 Minutes)</strong></td>
<td>CK12 Flexbook: Equations and Functions 1.1</td>
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<td>CK12 Flexbook: Equations and Functions 1.3</td>
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<td><strong>(40 Minutes)</strong></td>
<td>Data Plots Article: Determine meaning of words/phrases as they are used in text. Assignment (April 6 – 8)</td>
<td>Data Plots Article: Determine central idea and provide summary of a text.</td>
<td>Data Plots Article: Analyze how the author develops the text.</td>
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**Science (40 Minutes)**

**Online Learning Options:**
- Physical Science
  - Scientific Method Overview: Presentation and Questions
- Biology
  - Nature of Science: Read and answer questions
- Chemistry

**Physical Science**
- Methods of Science: Read and answer questions

**Biology**
- Methods of Science: Read and answer questions

**Chemistry**
- Methods of Science: Read and answer questions

**Physical Science**
- Standards of Measurement: Read and answer questions

**Biology**
- Standards of Measurement: Read and answer questions

**Chemistry**
- Standards of Measurement: Read and answer questions

**Physical Science**
- Communication with Graphs: Read and answer questions

**Biology**
- Using Graphs to Understand: Read and answer questions

**Chemistry**

**GOOD FRIDAY – NO ENRICHMENT ACTIVITIES TODAY**
<table>
<thead>
<tr>
<th>Date</th>
<th>Subject</th>
<th>Activity</th>
<th>Assignment/Questions/Video/Reading</th>
<th>Grade</th>
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<tbody>
<tr>
<td>GOOD FRIDAY</td>
<td></td>
<td>NO ENRICHMENT ACTIVITIES TODAY</td>
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<tr>
<td>Monday</td>
<td>Math</td>
<td>Check-Off Math</td>
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<tr>
<td>Monday</td>
<td>English</td>
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<td>Monday</td>
<td>Science</td>
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<td>Monday</td>
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<td>Tuesday</td>
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<tr>
<td>Friday</td>
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</table>
### Suggested Daily Schedule: Grades 9 - 12

<table>
<thead>
<tr>
<th>Time</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>8:00 – 9:00 am</td>
<td>Wake up, make your bed, eat breakfast and get ready for an awesome day!</td>
</tr>
<tr>
<td>9:00 – 9:40 am</td>
<td>Mathematics</td>
</tr>
<tr>
<td>9:40 – 10:20 am</td>
<td>English Language Arts</td>
</tr>
<tr>
<td>10:20 – 11:00 am</td>
<td>Science</td>
</tr>
<tr>
<td>11:00 – 12:30</td>
<td>Lunch, World Languages, and Free Time</td>
</tr>
<tr>
<td>12:30 – 1:10 pm</td>
<td>Social Studies</td>
</tr>
<tr>
<td>1:10 – 1:40</td>
<td>Afternoon Exercise</td>
</tr>
<tr>
<td>1:40 – 2:10</td>
<td>Current Events – watch the news or read the newspaper OR Language Acquisition</td>
</tr>
<tr>
<td>2:10-2:30</td>
<td>Social-Emotional Learning/Reflection/Organize for the Next Day</td>
</tr>
</tbody>
</table>

### Family Suggestions

<table>
<thead>
<tr>
<th><strong>Parent Suggestions</strong></th>
<th><strong>Student Suggestions</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>How can I support my student as a learner outside of school?</td>
<td>How can I continue learning outside of school?</td>
</tr>
<tr>
<td>□ Familiarize yourself with your child’s learning calendar.</td>
<td>□ Complete work on your suggested learning calendar.</td>
</tr>
<tr>
<td>□ Encourage your child to do their best when completing tasks and assignments.</td>
<td>□ Put in your best effort when completing tasks and assignments.</td>
</tr>
<tr>
<td>□ Contact your child’s teacher or the district’s homework hotline when you or your child have questions or need feedback.</td>
<td>□ Contact your teacher when you need help. Teachers are available via e-mail, your school’s online learning program or on the district’s homework hotline.</td>
</tr>
<tr>
<td>□ Support your child in starting the daily work early in the day. Waiting until the late afternoon or evening to start work adds unnecessary stress and creates missed opportunities for collaboration and feedback.</td>
<td>□ Let your teacher know if you have access to a phone or computer.</td>
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<tr>
<td>□ Remind your child to take frequent breaks to stay focused.</td>
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<tr>
<td>□ Consider designating a dedicated workspace to maximize time on task and facilitate learning.</td>
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<tr>
<td>□ Start your work early. Waiting until the late afternoon or evening to start work adds unnecessary stress and creates missed opportunities for collaboration and feedback.</td>
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<tr>
<td>□ Take short breaks to increase focus and stay motivated to complete tasks on time.</td>
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<tr>
<td>□ Find a quiet place to complete your work.</td>
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</tbody>
</table>
## Additional Student Supports

<table>
<thead>
<tr>
<th>Individual Supports</th>
<th>See “Individualizing Support for Students” for more information on how to provide additional support to your child while at home.</th>
</tr>
</thead>
</table>
| **English Language Learners** | **Enrichment Packet**  
- Daily language learning is important! The following links/resources are available for students to access daily language learning.  
- ¡El aprendizaje diario de idiomas es importante! Los siguientes enlaces/recursos están disponibles para que los estudiantes accedan al aprendizaje diario de idiomas.  
- Kujifunza lugha ya kila siku ni muhimu! Viungo vifuatavyo/rasilimali vinapatikana kwa wanafunzi kupata mafunzo ya lugha ya kila siku.  
- दैनिक भाषा सिक्ि महत्त्वपूर्ण छ! तलका लिङ्कहरू / स्रोतहरू विद्याय्हरूको लागि दैनिक भाषा सिक्िे पहुँचको लागि उपलब्ध छन्। |

| AP | College Board is offering free online courses on YouTube! Follow the link below to access their information.  
https://apstudents.collegeboard.org/coronavirus-updates |
# Contents

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  1.3 Solving Linear Systems by Linear Combinations (Elimination) ............ 47  
  1.4 Graphing and Solving Linear Inequalities ................................... 74  
  1.5 Solving Linear Systems in Three Variables .................................... 90  
  1.6 References .................................................................................. 99
Systems of Linear Equations and Inequalities

Chapter Outline

1.1 SOLVING LINEAR SYSTEMS BY GRAPHING
1.2 SOLVING LINEAR SYSTEMS BY SUBSTITUTION
1.3 SOLVING LINEAR SYSTEMS BY LINEAR COMBINATIONS (ELIMINATION)
1.4 GRAPHING AND SOLVING LINEAR INEQUALITIES
1.5 SOLVING LINEAR SYSTEMS IN THREE VARIABLES
1.6 REFERENCES

Systems of Linear Equations and Inequalities
This chapter of Algebra II is a review of solving linear systems that students learned in Algebra I. This concept is expanded to include solving systems of linear inequalities and solving systems of linear equations in three variables.
1.1 Solving Linear Systems by Graphing

Objective
Review the concept of the solution to a linear system in the context of a graph.

Review Queue
1. Graph the equation \( y = \frac{1}{2}x + 3 \).
2. Write the equation \( 4x - 3y = 6 \) in slope intercept form.
3. Solve for \( x \) in \( 7x - 3y = 26 \), given that \( y = 3 \).

Checking a Solution for a Linear System

Objective
Determine whether an ordered pair is a solution to a given system of linear equations.

Watch This

James Sousa: Ex: Identify the Solution to a System of Equations Given a Graph, Then Verify

Guidance
A system of linear equations consists of the equations of two lines. The solution to a system of linear equations is the point which lies on both lines. In other words, the solution is the point where the two lines intersect. To verify whether a point is a solution to a system or not, we will either determine whether it is the point of intersection of two lines on a graph (Example A) or we will determine whether or not the point lies on both lines algebraically (Example B).

Example A
Is the point (5, -2) the solution of the system of linear equations shown in the graph below?
Solution: Yes, the lines intersect at the point (5, -2) so it is the solution to the system.

Example B
Is the point (-3, 4) the solution to the system given below?

\[ 2x - 3y = -18 \]
\[ x + 2y = 6 \]

Solution: No, (-3, 4) is not the solution. If we replace the x and y in each equation with -3 and 4 respectively, only the first equation is true. The point is not on the second line; therefore it is not the solution to the system.

Guided Practice

1. Is the point (-2, 1) the solution to the system shown below?
2. Verify algebraically that (6, -1) is the solution to the system shown below.

\[
\begin{align*}
3x - 4y &= 22 \\
2x + 5y &= 7
\end{align*}
\]

3. Explain why the point (3, 7) is the solution to the system:

\[
\begin{align*}
y &= 7 \\
x &= 3
\end{align*}
\]

**Answers**

1. No, (-2, 1) is not the solution. The solution is where the two lines intersect which is the point (-3, 1).

2. By replacing \(x\) and \(y\) in both equations with 6 and -1 respectively (shown below), we can verify that the point (6, -1) satisfies both equations and thus lies on both lines.

\[
\begin{align*}
3(6) - 4(-1) &= 18 + 4 = 22 \\
2(6) + 5(-1) &= 12 - 5 = 7
\end{align*}
\]

3. The horizontal line is the line containing all points where the \(y\)-coordinate is 7. The vertical line is the line containing all points where the \(x\)-coordinate is 3. Thus, the point (3, 7) lies on both lines and is the solution to the system.

**Problem Set**

Match the solutions with their systems.

1. (1, 2)
2. (2, 1)
3. (-1, 2)
4. \((-1, -2)\)
Determine whether each ordered pair represents the solution to the given system.

5.

\[4x + 3y = 12\]
\[5x + 2y = 1; \ (-3, 8)\]

6.

\[3x - y = 17\]
\[2x + 3y = 5; \ (5, -2)\]
7.  
\[7x - 9y = 7\]
\[x + y = 1; \quad (1, 0)\]

8.  
\[x + y = -4\]
\[x - y = 4; \quad (5, -9)\]

9.  
\[x = 11\]
\[y = 10; \quad (11, 10)\]

10.  
\[x + 3y = 0\]
\[y = -5; \quad (15, -5)\]

11. Describe the solution to a system of linear equations.

12. Can you think of why a linear system of two equations would not have a unique solution?

---

### Solving Systems with One Solution Using Graphing

**Objective**

Graph lines to identify the unique solution to a system of linear equations.

**Watch This**

[Click image to the left or use the URL below.](http://www.ck12.org/flx/render/embeddedobject/60094)

**MEDIA**

Click image to the left or use the URL below.

URL: [http://www.ck12.org/flx/render/embeddedobject/60094](http://www.ck12.org/flx/render/embeddedobject/60094)

**James Sousa: Ex 1: Solve a System of Equations by Graphing**

**Guidance**

In this lesson we will be using various techniques to graph the pairs of lines in systems of linear equations to identify the point of intersection or the solution to the system. It is important to use graph paper and a straightedge to graph the lines accurately. Also, you are encouraged to check your answer algebraically as described in the previous lesson.

**Example A**

Graph and solve the system:
1.1. Solving Linear Systems by Graphing

\[
\begin{align*}
y &= -x + 1 \\
y &= \frac{1}{2}x - 2
\end{align*}
\]

**Solution:**

Since both of these equations are written in slope intercept form, we can graph them easily by plotting the \( y \)-intercept point and using the slope to locate additional points on each line.

The equation, \( y = -x + 1 \), graphed in **blue**, has \( y \)-intercept 1 and slope \(-\frac{1}{1}\).

The equation, \( y = \frac{1}{2}x - 2 \), graphed in **red**, has \( y \)-intercept -2 and slope \(\frac{1}{2}\).

Now that both lines have been graphed, the intersection is observed to be the point \((2, -1)\).

Check this solution algebraically by substituting the point into both equations.

**Example B**

Graph and solve the system:

\[
\begin{align*}
3x + 2y &= 6 \\
y &= -\frac{1}{2}x - 1
\end{align*}
\]

**Solution:** This example is very similar to the first example. The only difference is that equation 1 is not in slope intercept form. We can either solve for \( y \) to put it in slope intercept form or we can use the intercepts to graph the equation. To review using intercepts to graph lines, we will use the latter method.
Recall that the $x$–intercept can be found by replacing $y$ with zero and solving for $x$:

\[3x + 2(0) = 6\]
\[3x = 6\]
\[x = 2\]

Similarly, the $y$–intercept is found by replacing $x$ with zero and solving for $y$:

\[3(0) + 2y = 6\]
\[2y = 6\]
\[y = 3\]

We have two points, $(2, 0)$ and $(0, 3)$ to plot and graph this line. Equation 2 can be graphed using the $y$–intercept and slope as shown in Example A.

Now that both lines are graphed we observe that their intersection is the point $(4, -3)$.

Finally, check this solution by substituting it into each of the two equations.

Equation 1: $3x + 2y = 6; 3(4) + 2(-3) = 12 - 6 = 6 \checkmark$

Equation 2: $y = -\frac{1}{2}x - 1; -3 = -\frac{1}{2}(4) - 1 \checkmark$

**Example C**

In this example we will use technology to solve the system:

\[2x - 3y = 10\]
\[y = \frac{2}{3}x + 4\]
This process may vary somewhat based on the technology you use. All directions here can be applied to the TI-83 or 84 (plus, silver, etc) calculators.

**Solution:** The first step is to graph these equations on the calculator. The first equation must be rearranged into slope intercept form to put in the calculator.

\[
\begin{align*}
2x - 3y &= 10 \\
-3y &= -2x + 10 \\
y &= \frac{-2x + 10}{-3} \\
y &= \frac{2}{3}x - \frac{10}{3}
\end{align*}
\]

The graph obtained using the calculator should look like this:

The first equation, \( y = \frac{2}{3}x - \frac{10}{3} \), is graphed in blue. The second equation, \( y = -\frac{2}{3}x + 4 \), is graphed in red.

The solution does not lie on the “grid” and is therefore difficult to observe visually. With technology we can calculate the intersection. If you have a TI-83 or 84, use the CALC menu, select INTERSECT. Then select each line by pressing ENTER on each one. The calculator will give you a “guess.” Press ENTER one more time and the calculator will then calculate the intersection of \((5.5, .333...\)\). We can also write this point as \((\frac{11}{2}, \frac{1}{3})\). Check the solution algebraically.

Equation 1: \( 2x - 3y = 10; 2 \left( \frac{11}{2} \right) - 3 \left( \frac{1}{3} \right) = 11 - 1 = 10 \) \( \checkmark \)

Equation 2: \( y = -\frac{2}{3}x + 4; -\frac{2}{3} \left( \frac{11}{2} \right) + 4 = -\frac{11}{3} + \frac{12}{3} = \frac{1}{3} \) \( \checkmark \)

If you do not have a TI-83 or 84, the commands might be different. Check with your teacher.

**Guided Practice**

Solve the following systems by graphing. Use technology for problem 3.
1. 

\[ y = 3x - 4 \]
\[ y = 2 \]

2. 

\[ 2x - y = -4 \]
\[ 2x + 3y = -12 \]

3. 

\[ 5x + y = 10 \]
\[ y = \frac{2}{3}x - 7 \]

**Answers**

1. 

The first line is in slope intercept form and can be graphed accordingly.

The second line is a horizontal line through (0, 2).

The graph of the two equations is shown below. From this graph the solution appears to be (2, 2).

Checking this solution in each equation verifies that it is indeed correct.

Equation 1: \[ 2 = 3(2) - 4 \] ✓

Equation 2: \[ 2 = 2 \] ✓

2.
Neither of these equations is in slope intercept form. The easiest way to graph them is to find their intercepts as shown in Example B.

Equation 1: Let $y = 0$ to find the $x-$intercept.

\[
2x - y = -4 \\
2x - 0 = -4 \\
x = -2
\]

Now let $x = 0$, to find the $y-$intercept.

\[
2x - y = -4 \\
2(0) - y = -4 \\
y = 4
\]

Now we can use (-2, 0) and (0, 4) to graph the line as shown in the diagram. Using the same process, the intercepts for the second line can be found to be (-6, 0) and (0, -4).

Now the solution to the system can be observed to be (-3, -2). This solution can be verified algebraically as shown in the first problem.

3.
The first equation here must be rearranged to be $y = -5x + 10$ before it can be entered into the calculator. The second equation can be entered as is.

Using the calculate menu on the calculator the solution is $(3, -5)$.

Remember to verify this solution algebraically as a way to check your work.

**Problem Set**

Match the system of linear equations to its graph and state the solution.

1. 

   \[
   \begin{align*}
   3x + 2y &= -2 \\
   x - y &= -4
   \end{align*}
   \]
1.1. Solving Linear Systems by Graphing

b.

c.
2.

\[2x - y = 6\]
\[2x + 3y = 6\]
1.1. Solving Linear Systems by Graphing

b.

c.
3.

\[2x - 5y = -5\]
\[x + 5y = 5\]
4.

\[ y = 5x - 5 \]
\[ y = -x + 1 \]
1.1. Solving Linear Systems by Graphing

b.

c.
Solve the following linear systems by graphing. Use graph paper and a straightedge to insure accuracy. You are encouraged to verify your answer algebraically.

5.
\[
\begin{align*}
  y &= -\frac{2}{5}x + 1 \\
  y &= \frac{3}{5}x - 4
\end{align*}
\]

6.
\[
\begin{align*}
  y &= -\frac{2}{3}x + 4 \\
  y &= 3x - 7
\end{align*}
\]

7.
\[
\begin{align*}
  y &= -2x + 1 \\
  x - y &= -4
\end{align*}
\]

8.
\[
\begin{align*}
  3x + 4y &= 12 \\
  x - 4y &= 4
\end{align*}
\]

9.
\[
\begin{align*}
  7x - 2y &= -4 \\
  y &= -5
\end{align*}
\]
1.1. Solving Linear Systems by Graphing

10. 
\[ \begin{align*}
    x - 2y &= -8 \\
    x &= -3 
\end{align*} \]

Solve the following linear systems by graphing using technology. Solutions should be rounded to the nearest hundredth as necessary.

11. 
\[ \begin{align*}
    y &= \frac{3}{7}x + 11 \\
    y &= -\frac{13}{2}x - 5 
\end{align*} \]

12. 
\[ \begin{align*}
    y &= 0.95x - 8.3 \\
    2x + 9y &= 23 
\end{align*} \]

13. 
\[ \begin{align*}
    15x - y &= 22 \\
    3x + 8y &= 15 
\end{align*} \]

Use the following information to complete exercises 14-17.

Clara and her brother, Carl, are at the beach for vacation. They want to rent bikes to ride up and down the boardwalk. One rental shop, Bargain Bikes, advertises rates of $5 plus $1.50 per hour. A second shop, Frugal Wheels, advertises a rate of $6 plus $1.25 an hour.

14. How much does it cost to rent a bike for one hour from each shop? How about 10 hours?
15. Write equations to represent the cost of renting a bike from each shop. Let \( x \) represent the number of hours and \( y \) represent the total cost.
16. Solve your system to figure out when the cost is the same.
17. Clara and Carl want to rent the bikes for about 3 hours. Which shop should they use?

---

Solving Systems with No or Infinitely Many Solutions Using Graphing

**Objective**

Determine whether a system has a unique solution or not based on its graph. If no unique solution exists, determine whether there is no solution or infinitely many solutions.

**Guidance**

So far we have looked at linear systems of equations in which the lines always intersected in one, unique point. What happens if this is not the case? What could the graph of the two lines look like? In Examples A and B below we will explore the two possibilities.

**Example A**

Graph the system:
Solution:
In this example both lines have the same slope but different \( y \)-intercepts. When graphed, they are parallel lines and never intersect. This system has no solution. Another way to say this is to say that it is inconsistent.

Example B
Graph the system:

\[
\begin{align*}
2x - 3y &= 6 \\
-4x + 6y &= -12
\end{align*}
\]
Solution:
In this example both lines have the same slope and \( y \)-intercept. This is more apparent when the equations are written in slope intercept form:

\[
y = \frac{2}{3}x - 2 \quad \text{and} \quad y = \frac{2}{3}x - 2
\]

When we graph them, they are one line, coincident, meaning they have all points in common. This means that there are an infinite number of solutions to the system. Because this system has at least one solution it is considered to be consistent.

Consistent systems are systems which have at least one solution. If the system has exactly one, unique solution then it is independent. All of the systems we solved in the last section were independent. If the system has infinite solutions, like the system in Example B, then it is called dependent.

Example C
Classify the following system:

\[
\begin{align*}
10x - 2y &= 10 \\
y &= 5x - 5
\end{align*}
\]
Solution:
Rearranging the first equation into slope intercept form we get \( y = 5x - 5 \), which is exactly the same as the second equation. This means that they are the same line. Therefore the system is consistent and dependent.

Guided Practice
Classify the following systems as consistent, inconsistent, independent or dependent. You may do this with or without graphing them. You do not need to find the unique solution if there is one.

1. 
   
   \[
   \begin{align*}
   5x - y &= 15 \\
   x + 5y &= 15
   \end{align*}
   \]

2. 
   
   \[
   \begin{align*}
   9x - 12y &= -24 \\
   -3x + 4y &= 8
   \end{align*}
   \]

3. 
   
   \[
   \begin{align*}
   6x + 8y &= 12 \\
   -3x - 4y &= 10
   \end{align*}
   \]

Answers
1. The first step is to rearrange both equations into slope intercept form so that we can compare these attributes.

   \[
   \begin{align*}
   5x - y &= 15 \rightarrow y = 5x - 15 \\
   x + 5y &= 15 \rightarrow y = -\frac{1}{5}x + 3
   \end{align*}
   \]
1.1. Solving Linear Systems by Graphing

The slopes are not the same so the lines are neither parallel nor coincident. Therefore, the lines must intersect in one point. The system is consistent and independent.

2. Again, rearrange the equations into slope intercept form:

\[
9x - 12y = -24 \rightarrow y = \frac{3}{4}x + 2 \\
-3x + 4y = 8 \rightarrow y = \frac{3}{4}x + 2
\]

Now, we can see that both the slope and the \(y\)-intercept are the same and therefore the lines are coincident. The system is consistent and dependent.

3. The equations can be rewritten as follows:

\[
6x + 8y = 12 \rightarrow y = -\frac{3}{4}x + \frac{3}{2} \\
-3x - 4y = 10 \rightarrow y = -\frac{3}{4}x - \frac{5}{2}
\]

In this system the lines have the same slope but different \(y\)-intercepts so they are parallel lines. Therefore the system is inconsistent. There is no solution.

**Vocabulary**

**Parallel**

Two or more lines in the same plane that never intersect. They have the same slope and different \(y\)-intercepts.

**Coincident**

Lines which have all points in common. They are line which “coincide” with one another or are the same line.

**Consistent**

Describes a system with at least one solution.

**Inconsistent**

Describes a system with no solution.

**Dependent**

Describes a consistent system with infinite solutions.

**Independent**

Describes a consistent system with exactly one solution.

**Problem Set**

Describe the systems graphed below both algebraically (consistent, inconsistent, dependent, independent) and geometrically (intersecting lines, parallel lines, coincident lines).
Classify the following systems as consistent, inconsistent, independent or dependent. You may do this with or without graphing them. You do not need to find the unique solution if there is one.

4. \[4x - y = 8\]
\[y = 4x + 3\]

5. \[5x + y = 10\]
\[y = 5x + 10\]

6. \[2x - 2y = 11\]
\[y = x + 13\]

7. \[-7x + 3y = -21\]
\[14x - 6y = 42\]

8. \[y = -\frac{3}{5}x + 1\]
\[3x + 5y = 5\]

9. \[6x - y = 18\]
\[y = \frac{1}{6}x + 3\]
In problems 10-12 you will be writing your own systems. Your equations should be in standard form, $Ax + By = C$. Try to make them *look* different even if they are the same equation.

10. Write a system which is consistent and independent.
11. Write a system which is consistent and dependent.
12. Write a system which is inconsistent.
1.2 Solving Linear Systems by Substitution

Objective

To solve linear systems in two variables by using substitution to make an equation in one variable that can be solved.

Review Queue

1. Substitute \( x = 3 \) into the expression \( \frac{3(x-2)+5x}{x} \) and evaluate.
2. Solve for \( y \): \( 5x - 3y = 15 \)
3. Solve for \( x \): \( 2x + 14y = 42 \)

Solving Systems with One Solution Using Substitution

Objective

Solve consistent, independent systems using the substitution method.

Watch This

James Sousa: Ex 1: Solve a System of Equations Using Substitution

Guidance

In the substitution method we will be looking at the two equations and deciding which variable is easiest to solve for so that we can write one of the equations as \( x = \) or \( y = \). Next we will replace either the \( x \) or the \( y \) accordingly in the other equation. The result will be an equation with only one variable that we can solve.

Example A

Solve the system using substitution:

\[
\begin{align*}
2x + y &= 12 \\
-3x - 5y &= -11
\end{align*}
\]

Solution: The first step is to look for a variable that is easy to isolate. In other words, does one of the variables have a coefficient of 1? Yes, that variable is the \( y \) in the first equation. So, start by isolating or solving for \( y \): \( y = -2x + 12 \)

This expression can be used to replace the \( y \) in the other equation and solve for \( x \):
\[-3x - 5(-2x + 12) = -11\]
\[-3x + 10x - 60 = -11\]
\[7x - 60 = -11\]
\[7x = 49\]
\[x = 7\]

Now that we have found \(x\), we can use this value in our expression to find \(y\):

\[y = -2(7) + 12\]
\[y = -14 + 12\]
\[y = -2\]

Recall that the solution to a linear system is a point in the coordinate plane where the two lines intersect. Therefore, our answer should be written as a point: (7, -2). You can check your answer by substituting this point into both equations to make sure that it satisfies them:

\[2(7) + -2 = 14 - 2 = 12\]
\[-3(7) - 5(-2) = -21 + 10 = -11\]  \(✓\)

**Example B**

Solve the system using substitution:

\[2x + 3y = 13\]
\[x + 5y = -4\]

**Solution:** In the last example, \(y\) was the easiest variable to isolate. Is that the case here? No, this time, \(x\) is the variable with a coefficient of 1. It is easy to fall into the habit of always isolating \(y\) since you have done it so much to write equation in slope-intercept form. Try to avoid this and look at each system to see which variable is easiest to isolate. Doing so will help reduce your work.

Solving the second equation for \(x\) gives: \(x = -5y - 4\).

This expression can be used to replace the \(x\) in the other equation and solve for \(y\):

\[2(-5y - 4) + 3y = 13\]
\[-10y - 8 + 3y = 13\]
\[-7y - 8 = 13\]
\[-7y = 21\]
\[y = -3\]

Now that we have found \(y\), we can use this value in our expression to find \(x\):
1.2. Solving Linear Systems by Substitution

\[
x = -5(-3) - 4
\]
\[
x = 15 - 4
\]
\[
x = 11
\]

So, the solution to this system is (11, -3). Don’t forget to check your answer:

\[
2(11) + 3(-3) = 22 - 9 = 13
\]
\[
11 + 5(-3) = 11 - 15 = -4 \quad \checkmark
\]

**Example C**

Solve the system using substitution:

\[
4x + 3y = 4
\]
\[
6x - 2y = 19
\]

**Solution:** In this case, none of the variables have a coefficient of 1. So, we can just pick one to solve for. Let’s solve for the \( x \) in equation 1:

\[
4x = -3y + 4
\]
\[
x = -\frac{3}{4}y + 1
\]

Now, this expression can be used to replace the \( x \) in the other equation and solve for \( y \):

\[
6 \left( -\frac{3}{4}y + 1 \right) - 2y = 19
\]
\[
-\frac{18}{4}y + 6 - 2y = 19
\]
\[
-\frac{9}{2}y - \frac{4}{2}y = 13
\]
\[
-\frac{13}{2}y = 13
\]
\[
\left( -\frac{2}{13} \right) \left( -\frac{13}{2} \right) y = 13 \left( -\frac{2}{13} \right)
\]
\[
y = -2
\]

Now that we have found \( y \), we can use this value in our expression find \( x \):

\[
x = \left( -\frac{3}{4} \right)(-2) + 1
\]
\[
x = \frac{6}{4} + 1
\]
\[
x = \frac{3}{2} + \frac{2}{2}
\]
\[
x = \frac{5}{2}
\]
So, the solution is \( \left( \frac{5}{2}, -2 \right) \). Check your answer:

\[
4 \left( \frac{5}{2} \right) + 3(-2) = 10 - 6 = 4 \\
6 \left( \frac{5}{2} \right) - 2(-2) = 15 + 4 = 19 
\]

\( \checkmark \)

**Example D**

Rex and Carl are making a mixture in science class. They need to have 12 ounces of a 60% saline solution. To make this solution they have a 20% saline solution and an 80% saline solution. How many ounces of each do they need to make the correct mixture?

**Solution:** This type of word problem can be daunting for many students. Let’s try to make it easier by organizing our information into a “picture” equation as shown below:

In this picture, we can see that we will be mixing \( x \) ounces of the 20% solution with \( y \) ounces of the 80% solution to get 12 ounces of the 60% solution. The two equations are thus:

\[
0.2x + 0.8y = 0.6(12) \\
x + y = 12
\]

Now we can solve the system using substitution. Solve for \( y \) in the second equation to get: \( y = 12 - x \).

Now, substitute and solve in the first equation:

\[
0.2x + 0.8(12 - x) = 0.6(12) \\
0.2x + 9.6 - 0.8x = 7.2 \\
-0.6x = -2.4 \\
x = 4
\]

Now we can find \( y \):

\[
y = 12 - x \\
y = 12 - 4 \\
y = 8
\]
Therefore, Rex and Carl need 4 ounces of the 20% saline solution and 8 ounces of the 80% saline solution to make the correct mixture.

**Guided Practice**

Solve the following systems using the substitution method.

1. 

   \[3x + 4y = -13\]
   \[x = -2y - 9\]

2. 

   \[-2x - 5y = -39\]
   \[x + 3y = 24\]

3. 

   \[y = \frac{1}{2}x - 21\]
   \[y = -2x + 9\]

**Answers**

1. In this problem, the second equation is already solved for \(x\) so we can use that in the first equation to find \(y\):

   \[3(-2y - 9) + 4y = -13\]
   \[-6y - 27 + 4y = -13\]
   \[-2y - 27 = -13\]
   \[-2y = 14\]
   \[y = -7\]

   Now we can find \(x\):

   \[x = -2(-7) - 9\]
   \[x = 14 - 9\]
   \[x = 5\]

   Therefore the solution is \((5, -7)\).

2. This time the \(x\) in the second equation is the easiest variable to isolate: \(x = -3y + 24\). Let’s use this in the first expression to find \(y\):

   \[-2(-3y + 24) - 5y = -39\]
   \[6y - 48 - 5y = -39\]
   \[y = 9\]
Now we can find $x$:

\[
x = -3(9) + 24
\]
\[
x = -27 + 24
\]
\[
x = -3
\]

Therefore the solution is (-3, 9).

3. In this case, both equations are equal to $y$. Since $y = y$, by the Reflexive Property of Equality, we can let the right hand sides of the equations be equal too. This is still a substitution problem; it just looks a little different.

\[
\frac{1}{2}x - 21 = -2x + 9
\]
\[
2 \left( \frac{1}{2}x - 21 = -2x + 9 \right)
\]
\[
x - 42 = -4x + 18
\]
\[
5x = 60
\]
\[
x = 12
\]

Now we can find $y$:

\[
y = \frac{1}{2}(12) - 21 \quad y = -2(12) + 9
\]
\[
y = 6 - 21 \quad or \quad y = -24 + 9
\]
\[
y = -15 \quad y = -15
\]

Therefore our solution is (12, -15).

**Problem Set**

Solve the following systems using substitution. Remember to check your answers.

1.

\[
x + 3y = -1
\]
\[
2x + 9y = 7
\]

2.

\[
7x + y = 6
\]
\[
x - 2y = -12
\]

3.

\[
5x + 2y = 0
\]
\[
y = x - 7
\]
Solving Linear Systems by Substitution

1.2. Solving Linear Systems by Substitution

4. 
\[2x - 5y = 21\]
\[x = -6y + 2\]

5. 
\[y = x + 3\]
\[y = 2x - 1\]

6. 
\[x + 6y = 1\]
\[-2x - 11y = -4\]

7. 
\[2x + y = 18\]
\[-3x + 11y = -27\]

8. 
\[2x + 3y = 5\]
\[5x + 7y = 8\]

9. 
\[-7x + 2y = 9\]
\[5x - 3y = 3\]

10. 
\[2x - 6y = -16\]
\[-6x + 10y = 8\]

11. 
\[2x - 3y = -3\]
\[8x + 6y = 12\]

12. 
\[5x + y = -3\]
\[y = 15x + 9\]

Set up and solve a system of linear equations to answer each of the following word problems.

13. Alicia and Sarah are at the supermarket. Alicia wants to get peanuts from the bulk food bins and Sarah wants to get almonds. The almonds cost $6.50 per pound and the peanuts cost $3.50 per pound. Together they buy 1.5 pounds of nuts. If the total cost is $6.75, how much did each girl get? Set up a system to solve using substitution.

14. Marcus goes to the department store to buy some new clothes. He sees a sale on t-shirts ($5.25) and shorts ($7.50). Marcus buys seven items and his total, before sales tax, is $43.50. How many of each item did he buy?

15. Jillian is selling tickets for the school play. Student tickets are $3 and adult tickets are $5. If 830 people buy tickets and the total revenue is $3104, how many students attended the play?
Solving Systems with No or Infinitely Many Solutions Using Substitution

**Objective**

To understand how a system with no solution and a system with infinitely many solutions are discovered using the substitution method.

**Guidance**

When a system has no solution or an infinite number of solutions and we attempt to find a single, unique solution using an algebraic method, such as substitution, the variables will cancel out and we will have an equation consisting of only constants. If the equation is untrue as seen below in Example A, then the system has no solution. If the equation is always true, as seen in Example B, then there are infinitely many solutions.

**Example A**

Solve the system using substitution:

\[
\begin{align*}
3x - 2y &= 7 \\
y &= \frac{3}{2}x + 5
\end{align*}
\]

**Solution:** Since the second equation is already solved for \( y \), we can use this in the first equation to solve for \( x \):

\[
3x - 2 \left( \frac{3}{2}x + 5 \right) = 7
\]

\[
3x - 3x + 5 = 7
\]

\[
5 \neq 7
\]

Since the substitution above resulted in the elimination of the variable, \( x \), and an untrue equation involving only constants, the system has no solution. The lines are parallel and the system is inconsistent.

**Example B**

Solve the system using substitution:

\[
\begin{align*}
-2x + 5y &= -2 \\
4x - 10y &= 4
\end{align*}
\]

**Solution:** We can solve for \( x \) in the first equation as follows:

\[
-2x = -5y - 2
\]

\[
x = \frac{5}{2}y + 1
\]

Now, substitute this expression into the second equation and solve for \( y \):
In the process of solving for \( y \), the variable is cancelled out and we are left with only constants. We can stop at the step where \( 4 = 4 \) or continue and subtract 4 on each side to get \( 0 = 0 \). Either way, this is a true statement. As a result, we can conclude that this system has an infinite number of solutions. The lines are coincident and the system is consistent and dependent.

**Guided Practice**

Solve the following systems using substitution. If there is no unique solution, state whether there is no solution or infinitely many solutions.

1. 
   \[
   \begin{align*}
   y &= \frac{2}{5}x - 3 \\
   2x - 5y &= 15
   \end{align*}
   \]

2. 
   \[
   \begin{align*}
   -x + 7y &= 5 \\
   3x - 21y &= -5
   \end{align*}
   \]

3. 
   \[
   \begin{align*}
   3x - 5y &= 0 \\
   -2x + 6y &= 0
   \end{align*}
   \]

**Answers**

1. Substitute the first equation into the second and solve for \( x \):

   \[
   \begin{align*}
   2x - 5\left(\frac{2}{5}x - 3\right) &= 15 \\
   2x - 2x + 15 &= 15 \\
   15 &= 15 \\
   (0 &= 0)
   \end{align*}
   \]

Since the result is a true equation, the system has infinitely many solutions.

2. Solve the first equation for \( x \) to get: \( x = 7y - 5 \). Now, substitute this into the second equation to solve for \( y \):

   \[
   \begin{align*}
   3(7y - 5) - 21y &= -5 \\
   21y - 15 - 21y &= -5 \\
   -15 &\neq -5
   \end{align*}
   \]
Since the result is an untrue equation, the system has no solution.

3. Solving the second equation for $x$ we get: $x = 3y$. Now, we can substitute this into the first equation to solve for $y$:

$3(3y) - 5y = 0$
$9y - 5y = 0$
$4y = 0$
$y = 0$

Now we can use this value of $y$ to find $x$:

$x = 3y$
$x = 3(0)$
$x = 0$

Therefore, this system has a solution at (0, 0). After solving systems that result in $0 = 0$, it is easy to get confused by a result with zeros for the variables. It is perfectly okay for the intersection of two lines to occur at (0, 0).

**Problem Set**

Solve the following systems using substitution.

1. 

$17x - 3y = 5$
$y = 3x + 1$

2. 

$4x - 14y = 21$
$y = \frac{2}{7}x + 7$

3. 

$-24x + 9y = 12$
$8x - 3y = -4$

4. 

$y = -\frac{3}{4}x + 9$
$6x + 8y = 72$

5. 

$2x + 7y = 12$
$y = -\frac{2}{3}x + 4$
6. 
\[ 2x = -6y + 11 \]
\[ y = -\frac{1}{3}x + 7 \]

7. 
\[ \frac{1}{2}x - \frac{4}{5}y = 8 \]
\[ 5x - 8y = 50 \]

8. 
\[ -6x + 16y = 38 \]
\[ x = \frac{8}{3}y - \frac{19}{3} \]

9. 
\[ x = y \]
\[ 5x + 3y = 0 \]

10. 
\[ \frac{1}{2}x + 3y = -15 \]
\[ y = x - 5 \]

11. 
\[ \frac{2}{3}x + \frac{1}{6}y = 2 \]
\[ y = -4x + 12 \]

12. 
\[ 16x - 4y = 3 \]
\[ y = 4x + 7 \]
1.3 Solving Linear Systems by Linear Combinations (Elimination)

Objective
Solve systems by combining the two equations (or multiples of the two equations) to eliminate a variable.

Review Queue
1. Solve the system by graphing:

\[ y = \frac{2}{3}x - 3 \]
\[ y = -\frac{1}{2}x + 4 \]

2. Solve the system by substitution:

\[ x + 3y = 2 \]
\[ 4x + y = -25 \]

3. Find the least common multiple of the following pairs of numbers:
   a) 3 and 5
   b) 2 and 10
   c) 6 and 15
   d) 4 and 10

Solving Systems Without Multiplying

Objective
Solve systems by adding the two equations together to eliminate a variable.

Watch This

Khan Academy: Addition Elimination Method I

Guidance
In this lesson we will be looking at systems in which the two equations contain coefficients of one variable that are additive inverses (opposites) of one another.

**Example A**
Solve the system using Linear Combination:

\[
\begin{align*}
2x - 3y &= -9 \\
5x + 3y &= 30
\end{align*}
\]

**Solution:** Notice that the coefficients of the \( y \) terms are opposites. When we add the two equations together, these terms will be eliminated because their sum is \( 0y = 0 \).

\[
\begin{align*}
2x - 3y &= -9 \\
+ 5x + 3y &= 30 \\
7x &= 21
\end{align*}
\]

Now we can solve for \( x \):

\[
7x = 21 \\
x = 3
\]

Now that we have found \( x \), we can plug this value into either equation to find \( y \):

\[
\begin{align*}
2(3) - 3y &= -9 \\
6 - 3y &= -9 & 5(3) + 3y &= 30 \\
-3y &= -15 & 15 + 3y &= 30 \\
y &= 5 & or & 3y &= 15 \\
& & y &= 5
\end{align*}
\]

The solution is therefore: \((3, 5)\).

Remember to check your answer:

\[
\begin{align*}
2(3) - 3(5) &= 6 - 15 = -9 \\
5(3) + 3(5) &= 15 + 15 = 30 \checkmark
\end{align*}
\]

**Example B**
Solve the system using Linear Combination:

\[
\begin{align*}
x + 4y &= 2 \\
x - 5y &= -3
\end{align*}
\]

**Solution:** Notice that the coefficients of the \( x \) terms are opposites. When we add the two equations together, these terms will be eliminated because their sum is \( 0x = 0 \).
\[ x + 4y = 2 \\
-\ x - 5y = -3 \\
\hline
\]
\[-y = -1 \\
y = 1\]

Now we can solve for \( y \):

\[-y = -1 \\
y = 1\]

Now that we have found \( y \), we can plug this value into either equation to find \( x \):

\[
x + 4(1) = 2 \\
x + 4 = 2 \\
x = -2 \\
\]

or

\[
-x - 5(1) = -3 \\
x - 5 = 3 \\
x = 2 \\
\]

The solution is therefore: (-2, 1).

Remember to check your answer:

\[
-2 + 4(1) = -2 + 4 = 2 \\
-(\ -2\ ) - 5(1) = 2 - 5 = 3 \quad \checkmark
\]

**Example C**

Solve the system using Linear Combination:

\[
2x + y = 2 \\
-3x + y = -18
\]

**Solution:** In this case the coefficients of the \( y \) terms are the same, not opposites. One way to solve this system using linear combination would be to subtract the second equation from the first instead of adding it. Sometimes subtraction results in more errors, however, particularly when negative numbers are involved. Instead of subtracting, multiply the second equation by -1 and then add them together.

\[
-1(-3x + y = -18) \\
3x - y = 18
\]

Essentially, we changed all of the signs of the terms in this equation.

Now we can add the two equations together to eliminate \( y \):
1.3. Solving Linear Systems by Linear Combinations (Elimination)

\[ \begin{align*}
2x + y &= 2 \\
+ 3x - y &= 18 \\
5x &= 20
\end{align*} \]

Now we can solve for \( x \):

\[ \begin{align*}
5x &= 20 \\
x &= 4
\end{align*} \]

Now that we have found \( x \), plug this value into either equation to find \( y \):

\[ \begin{align*}
2(4) + y &= 2 \\
8 + y &= 2 \quad \text{or} \quad -3(4) + y = -18 \\
8 + y &= 2 \\
y &= -6
\end{align*} \]

The solution is therefore: \( (4, -6) \).

Remember to check your answer:

\[ \begin{align*}
2(4) + (-6) &= 8 - 6 = 2 \quad \checkmark \\
-3(4) + (-6) &= -12 - 6 = -18
\end{align*} \]

**Guided Practice**

Solve the following systems using Linear Combinations.

1. \[ \begin{align*}
4x + 5y &= 8 \\
-2x - 5y &= 6
\end{align*} \]

2. \[ \begin{align*}
2x + 3y &= 3 \\
2x - y &= 23
\end{align*} \]

3. \[ \begin{align*}
2x + 3y &= -6 \\
y &= 2x - 2
\end{align*} \]

**Answers**

1. First we can add the two equations together to eliminate \( y \) and solve for \( x \):
Substitute $x$ into one equation to find $y$:

$$
4(7) + 5y = 8 \\
28 + 5y = 8 \\
5y = -20 \\
y = -4
$$

Solution: (7, -4)

2. This time we need to begin by multiplying the second equation by -1 to get $-2x + y = -23$. Now we can add the two equations together to eliminate $x$ and solve for $y$:

$$
2x + 3y = 3 \\
+ (-2x + y = -23) \\
4y = -20 \\
y = -5
$$

Substitute $y$ into one equation to find $x$:

$$
2x + (3 - 5) = 3 \\
2x - 15 = 3 \\
2x = 18 \\
x = 9
$$

Solution: (9, -5)

3. In this example, the second equation is not written in standard form. We must first rewrite this equation in standard form so that the variable will align vertically when we add the equations together. The second equation should be $-2x + y = -2$ after we subtract $2x$ from both sides. Now we can add the two equations together to eliminate $x$ and solve for $y$:

$$
2x + 3y = -6 \\
+ (-2x + y = -2) \\
4y = -8 \\
y = -2
$$

Substitute $y$ into one equation to find $x$: 
1.3. Solving Linear Systems by Linear Combinations (Elimination)

\[ 2x + 3(-2) = -6 \]
\[ 2x - 6 = -6 \]
\[ 2x = 0 \]
\[ x = 0 \]

Solution: (0, -2)

Vocabulary

Additive Inverse (Opposite)

The additive inverse of a number is the number multiplied by -1. When a number and it’s additive inverse are added together the result is zero (the identity for addition). Examples of opposites: 2 and -2, -3 and 3, 15 and -15, \( \frac{1}{2} \) and \( -\frac{1}{2} \).

Problem Set

Solve the following systems using linear combinations.

1. 
   \[ 4x + 2y = -6 \]
   \[ -5x - 2y = 4 \]

2. 
   \[ -3x + 5y = -34 \]
   \[ 3x - y = 14 \]

3. 
   \[ x + y = -1 \]
   \[ x - y = 21 \]

4. 
   \[ 2x + 8y = -4 \]
   \[ -2x + 3y = 15 \]

5. 
   \[ 8x - 12y = 24 \]
   \[ -3x + 12y = 21 \]

6. 
   \[ x + 3y = -2 \]
   \[ -x - 2y = 4 \]
7. 
\[ \begin{align*} 
5x + 7y &= 2 \\
5x + 3y &= 38 
\end{align*} \]

8. 
\[ \begin{align*} 
12x - 2y &= 2 \\
5x - 2y &= -5 
\end{align*} \]

9. 
\[ \begin{align*} 
2x + y &= 25 \\
x + y &= 5 
\end{align*} \]

10. 
\[ \begin{align*} 
\frac{1}{2}x + 3y &= -3 \\
y &= \frac{1}{2}x - 5 
\end{align*} \]

11. 
\[ \begin{align*} 
3x + 5y &= 10 \\
y &= -3x - 10 
\end{align*} \]

12. 
\[ \begin{align*} 
6x + 3y &= -3 \\
3y &= -7x + 1 
\end{align*} \]

13. 
\[ \begin{align*} 
4x - 2y &= 5 \\
-4x + 2y &= 11 
\end{align*} \]

14. 
\[ \begin{align*} 
9x + 2y &= 0 \\
-9x - 3y &= 0 
\end{align*} \]

15. 
\[ \begin{align*} 
11x + 7y &= 12 \\
-11x &= 7y - 12 
\end{align*} \]

Set up and solve a linear system of equations to solve the following word problems.

16. The sum of two numbers is 15 and their difference is 3. Find the two numbers.
17. Jessica and Maria got to the supermarket to buy fruit. Jessica buys 5 apples and 6 oranges and her total before tax is $3.05. Maria buys 7 apples and 6 oranges and her total before tax is $3.55. What is the price of each fruit? Hint: Let \( x \) be the price of one apple and \( y \) be the price of one orange.
1.3. Solving Linear Systems by Linear Combinations (Elimination)

Solving Systems by Multiplying One Equation to Cancel a Variable

Objective
Solve systems using linear combinations by multiplying one equation by a constant to eliminate a variable.

Watch This
Khan Academy: Solving systems by elimination2

Guidance
It is not always the case that the coefficients of one variable in a system of linear equations are the same or opposites. When this is not the case, we may be able to multiply one of the equations by a constant so that we have opposite coefficients of one variable and can proceed to solve the system as we have previously done.

Example A
Solve the system using linear combinations:

\[
\begin{align*}
4x + y &= 0 \\
x - 3y &= 26
\end{align*}
\]

Solution: Here we have coefficients of \( y \) that are opposite signs (one is positive and one is negative). We can get opposite values if we multiply the first equation by 3. Be careful; make sure you multiply the entire equation, including the constant, by 3:

\[
\begin{align*}
3(4x + y) &= 0 \\
12x + 3y &= 0
\end{align*}
\]

Now we can use this new equation in our system to eliminate \( y \) and solve for \( x \):

\[
\begin{align*}
12x + 3y &= 0 \\
+ x - 3y &= 26 \\
13x &= 26 \\
x &= 2
\end{align*}
\]

Now, find \( y \):

\[
\begin{align*}
4(2) + y &= 0 \\
8 + y &= 0 \\
y &= -8
\end{align*}
\]
Solution: (2, -8)

Check your answer:

\[
\begin{align*}
4(2) + (-8) &= 8 - 8 = 0 \\
(2) - 3(-8) &= 2 + 24 = 26 \quad \checkmark
\end{align*}
\]

* Note that we could have used the other equation to find \( y \) in the final step.

** From the beginning, we could have multiplied the second equation by -4 instead to cancel out the \( x \) variable.

Example B

Solve the system using linear combinations:

\[
\begin{align*}
2x + 5y &= 1 \\
y &= -3x + 21
\end{align*}
\]

Solution: For this system, we must first rewrite the second equation in standard form so that we can see how the coefficients compare. If we add 3\( x \) to both sides we get:

\[
\begin{align*}
2x + 5y &= 1 \\
3x + y &= 21
\end{align*}
\]

Now, we can see that if we multiply the second equation by -5, the coefficients of \( y \) will be opposites.

\[
\begin{align*}
2x + 5y &= 1 \\
-15x - 5y &= -105 \\
\hline
-13x &= -104 \\
x &= 8
\end{align*}
\]

Now, find \( y \):

\[
\begin{align*}
y &= -3(8) + 21 \\
y &= -24 + 21 \\
y &= -3
\end{align*}
\]

Solution: (8, -3)

Check your answer:

\[
\begin{align*}
2(8) + 5(-3) &= 16 - 15 = 1 \\
-3 &= -3(8) + 21 = -24 + 21 = -3 \quad \checkmark
\end{align*}
\]
Example C
Solve the system using linear combinations:

\[
\begin{align*}
4x - 6y &= -12 \\
y &= \frac{2}{3}x + 2
\end{align*}
\]

Solution: Again, we need to rearrange the second equation in this system to get it in standard form. We can do this by subtracting \(\frac{2}{3}x\) on both sides to get the following system:

\[
\begin{align*}
4x - 6y &= -12 \\
-\frac{2}{3}x + y &= 2
\end{align*}
\]

Multiply the second equation by 6 to eliminate \(y\):

\[
6 \left(-\frac{2}{3}x + y = 2\right) \\
-4x + 6y = 12
\]

And add it to the first equation.

\[
\begin{align*}
4x - 6y &= -12 \\
-4x + 6y &= 12
\end{align*}
\]

\[
0x = 0 \\
0 = 0
\]

Here, both variables were eliminated and we wound up with \(0 = 0\). Recall that this is a true statement and thus this system has infinite solutions.

Guided Practice
Solve the following systems using linear combinations.

1. 
\[
\begin{align*}
3x + 12y &= -3 \\
-x - 5y &= 0
\end{align*}
\]

2. 
\[
\begin{align*}
0.75x + 5y &= 0 \\
0.25x - 9y &= 0
\end{align*}
\]

3. 
\[
\begin{align*}
x - 3y &= 5 \\
y &= \frac{1}{3}x + 8
\end{align*}
\]
Answers

1. In this problem we can just multiply the second equation by 3 to get coefficients of $x$ which are opposites: $3(-x - 5y = 0) \Rightarrow -3x - 15y = 0$

\[
\begin{align*}
3x + 12y &= -3 \\
-3x - 15y &= 0
\end{align*}
\]

Now we can find $x$:

\[
\begin{align*}
3x + 12(1) &= -3 \\
3x + 12 &= -3 \\
3x &= -15 \\
x &= -5
\end{align*}
\]

Solution: (-5, 1)

2. For this system, we need to multiply the second equation by -3 to get coefficients of $x$ which are opposites: $-3(0.25x - 9y = 0) \Rightarrow -0.75x + 27y = 0$

\[
\begin{align*}
0.75x + 5y &= 0 \\
-0.75x - 9y &= 0
\end{align*}
\]

Now we can find $x$:

\[
\begin{align*}
0.75x + 5(0) &= 0 \\
0.75x &= 0 \\
x &= 0
\end{align*}
\]

Solution: (0, 0)

3. This time we need to rewrite the second equation in standard form:

\[
\begin{align*}
x - 3y &= 5 \\
-\frac{1}{3}x + y &= 8
\end{align*}
\]

Now we can multiply the second equation by 3 to get coefficients of $x$ that are opposites: $3 \left( -\frac{1}{3}x + y = 8 \right) \Rightarrow -x + 3y = 24$, Now our system is:
When we add these equations together, both variables are eliminated and the result is $0 = 29$, which is an untrue statement. Therefore, this system has no solution.

Problem Set

Solve the following systems using linear combinations.

1. 
   \[ \begin{align*}
   x - 7y &= 27 \\
   2x + y &= 9
   \end{align*} \]

2. 
   \[ \begin{align*}
   x + 3y &= 31 \\
   3x - 5y &= -5
   \end{align*} \]

3. 
   \[ \begin{align*}
   10x + y &= -6 \\
   -7x - 5y &= -13
   \end{align*} \]

4. 
   \[ \begin{align*}
   2x + 4y &= 18 \\
   x - 5y &= 9
   \end{align*} \]

5. 
   \[ \begin{align*}
   2x + 6y &= 8 \\
   3x + 2y &= -23
   \end{align*} \]

6. 
   \[ \begin{align*}
   12x - y &= 2 \\
   2x + 5y &= 21
   \end{align*} \]

7. 
   \[ \begin{align*}
   2x + 4y &= 24 \\
   -3x - 2y &= -26
   \end{align*} \]

8. 
   \[ \begin{align*}
   3x + 2y &= 19 \\
   5x + 4y &= 23
   \end{align*} \]
9.
\begin{align*}
3x - 9y &= 13 \\
x - 3y &= 7
\end{align*}

10.
\begin{align*}
8x + 2y &= -4 \\
3y &= -16x + 2
\end{align*}

11.
\begin{align*}
3x + 2y &= -3 \\
-6x - 5y &= 4
\end{align*}

12.
\begin{align*}
10x + 6y &= -24 \\
y &= -\frac{5}{3}x - 4
\end{align*}

13.
\begin{align*}
\frac{1}{3}x - \frac{2}{3}y &= -8 \\
\frac{1}{2}x - \frac{1}{3}y &= 12
\end{align*}

14.
\begin{align*}
6x - 10y &= -8 \\
y &= -\frac{3}{5}x
\end{align*}

15.
\begin{align*}
4x - 14y &= -52 \\
y &= \frac{2}{7}x + 3
\end{align*}

Set up and solve a system of linear equations for each of the following word problems.

16. Lia is making a mixture of Chlorine and water in her science class. She needs to make 13 ml of a 60% chlorine solution from a solution that is 35% chlorine and a second solution which is 75% chlorine. How many milliliters of each solution does she need?

17. Chelsea and Roberto each sell baked goods for their club’s fundraiser. Chelsea sells 13 cookies and 7 brownies and collects a total of $11.75. Roberto sells 10 cookies and 14 brownies and collects a total of $15.50. How much did they charge for the cookies and the brownies?

18. Mattie wants to plant some flowers in her yard. She has space for 15 plants. She buys pansies and daisies at her local garden center. The pansies are each $2.75 and the daisies are each $2.00. How many of each does she buy if she spends a total of $35.25?
Solving Systems by Multiplying Both Equations to Cancel a Variable

Objective
Solve linear systems using linear combinations in which both equations must be multiplied by a constant to cancel a variable.

Watch This
Watch the second portion of this video, starting around 5:00.

Khan Academy: Solving Systems of Equations by Multiplication

Guidance
In the linear systems in this lesson, we will need to multiply both equations by a constant in order to have opposite coefficients of one of the variables. In order to determine what numbers to multiply by, we will be finding the least common multiple of the given coefficients. Recall that the least common multiple of two numbers is the smallest number which is divisible by both of the given numbers. For example, 12 is the least common multiple of 4 and 6 because it is the smallest number divisible by both 4 and 6.

Example A
Solve the system using linear combinations:

\[
\begin{align*}
2x - 5y &= 15 \\
3x + 7y &= 8
\end{align*}
\]

Solution: In this problem we cannot simply multiply one equation by a constant to get opposite coefficients for one of the variables. Here we will need to identify the least common multiple of the coefficients of one of the variables and use this value to determine what to multiply each equation by. If we look at the coefficients of \(x\), 2 and 3, the least common multiple of these numbers is 6. So, we want to have the coefficients of \(x\) be 6 and -6 so that they are opposites and will cancel when we add the two equations together. In order to get coefficients of 6 and -6 we can multiply the first equation by 3 and the second equation by -2 (it doesn’t matter which one we make negative.)

\[
\begin{align*}
3(2x - 5y &= 15) &\Rightarrow & 6x - 15y &= 45 \\
-2(3x + 7y &= 8) &\Rightarrow & -6x - 14y &= -16 \\
\hline \\
& & & -29y &= 29 \\
& & & y &= -1
\end{align*}
\]

Now find \(x\):
\[ 2(x) - 5(-1) = 15 \]
\[ 2x + 5 = 15 \]
\[ 2x = 10 \]
\[ x = 5 \]

Solution: (5, -1)

Check your answer:

\[ 2(5) - 5(-1) = 10 + 5 = 15 \]
\[ 3(5) + 7(-1) = 15 - 7 = 8 \]

\[ \checkmark \]

* This problem could also be solved by eliminating the \( y \) variables first. To do this, find the least common multiple of the coefficients of \( y \), 5 and 7. The least common multiple is 35, so we would multiply the first equation by 7 and the second equation by 5. Since one of them is already negative, we don’t have to multiply by a negative.

**Example B**

Solve the system using linear combinations:

\[ 7x + 20y = -9 \]
\[ -2x - 3y = 8 \]

**Solution:** The first step is to decide which variable to eliminate. Either one can be eliminated but sometimes it is helpful to look at what we need to multiply by to eliminate each one and determine which is easier to eliminate. In general, it is easier to work with smaller numbers so in this case it makes sense to eliminate \( x \) first. The Least Common Multiple (LCM) of 7 and 2 is 14. To get 14 as the coefficient of each term, we need to multiple the first equation by 2 and the second equation by 7:

\[ 2(7x + 20y = -9) \Rightarrow 14x + 40y = -18 \]
\[ 7(-2x - 3y = 8) \]

\[ 14x - 21y = 56 \]

\[ 19y = 38 \]
\[ y = 2 \]

Now find \( x \):

\[ -2x - 3(2) = 8 \]
\[ -2x - 6 = 8 \]
\[ -2x = 14 \]
\[ x = -7 \]

Solution: (-7, 2)

Check your answer:
Example C
Solve the system using linear combinations:

\[
\begin{align*}
14x - 6y &= -3 \\
16x - 9y &= -7
\end{align*}
\]

**Solution:** This time, we will eliminate \( y \). We need to find the LCM of 6 and 9. The LCM is 18, so we will multiply the first equation by 3 and the second equation by -2. Again, it doesn’t matter which equation we multiply by a negative value.

\[
\begin{align*}
3(14x - 6y &= -3) &\Rightarrow +42x - 18y = -9 \\
-2(16x - 9y &= -7) &\Rightarrow -32x + 18y = 14
\end{align*}
\]

\[
\begin{align*}
10x &= 5 \\
x &= 2
\end{align*}
\]

Now find \( y \):

\[
\begin{align*}
14 \left( \frac{1}{2} \right) - 6y &= -3 \\
7 - 6y &= -3 \\
-6y &= -10 \\
y &= \frac{10}{6} = \frac{5}{3}
\end{align*}
\]

Solution: \( \left( \frac{1}{2}, \frac{5}{3} \right) \)

Check your answer:

\[
\begin{align*}
14 \left( \frac{1}{2} \right) - 6 \left( \frac{5}{3} \right) &= 7 - 10 = -3 \\
16 \left( \frac{1}{2} \right) - 9 \left( \frac{5}{3} \right) &= 8 - 15 = -7 \quad \checkmark
\end{align*}
\]

Example D
A one pound mix consisting of 30% cashews and 70% pistachios sells for $6.25. A one pound mix consisting of 80% cashews and 20% pistachios sells for $7.50. How would a mix consisting of 50% of each type of nut sell for?

**Solution:** First we need to write a system of linear equations to represent the given information. Let \( x \) = the cost of the cashews per pound and let \( y \) = the cost of the pistachios per pound. Now we can write two equations to represent the two different mixes of nuts:
Now we can solve this system to determine the cost of each type of nut per pound. If we eliminate $y$, we will need to multiple the first equation by 2 and the second equation by -7:

\[
\begin{align*}
2(0.3x - 0.7y &= 6.25) \Rightarrow 0.6x - 14y = 12.5 \\
-7(0.8x + 0.2y &= 7.50) \Rightarrow -5.6x + 14y = -52.5 \\
\hline
-5x &= -40 \\
x &= 8
\end{align*}
\]

Now find $y$:

\[
\begin{align*}
0.3(8) + 0.7y &= 6.25 \\
2.4 + 0.7y &= 6.25 \\
0.7y &= 3.85 \\
y &= 5.5
\end{align*}
\]

So, we have determined that the cost of the cashews is $8 per pound and the cost of the pistachios is $5.50 per pound. Now we can determine the cost of the 50% mix as follows:

\[
0.5(8.00) + 0.5(5.50) = 4.00 + 2.25 = 6.25 \text{ So, the new mix is } $6.25 \text{ per pound.}
\]

**Guided Practice**

Solve the following systems using linear combinations:

1.

\[
\begin{align*}
6x + 5y &= 3 \\
-4x - 2y &= -14
\end{align*}
\]

2.

\[
\begin{align*}
9x - 7y &= -19 \\
5x - 3y &= -15
\end{align*}
\]

3.

\[
\begin{align*}
15x - 21y &= -63 \\
7y &= 5x + 21
\end{align*}
\]

**Answers**

1. We can eliminate either variable here. To eliminate $x$, we can multiple the first equations by 2 and the second equation by 3 to get 12 - the LCM of 6 and 4.
1.3. Solving Linear Systems by Linear Combinations (Elimination)

\[ 2(6x + 5y = 3) \Rightarrow 12x + 10y = 6 \]
\[ 3(-4x - 2y = -14) \Rightarrow -12x - 6y = -42 \]
\[ \begin{align*}
4y &= -36 \\
y &= -9
\end{align*} \]

Now find \( x \):

\[ 6x + 5(-9) = 3 \]
\[ 6x - 45 = 3 \]
\[ 6x = 48 \]
\[ x = 8 \]

Solution: \((8, -9)\)

2. Again we can eliminate either variable. To eliminate \( y \), we can multiply the first equation by 3 and the second equation by -7:

\[ 3(9x - 7y = -19) \Rightarrow +27x - 21y = -57 \]
\[ -7(5x - 3y = -15) \Rightarrow -35x + 21y = 105 \]
\[ \begin{align*}
-8x &= 48 \\
x &= -6
\end{align*} \]

Now find \( y \):

\[ 5(-6) - 3y = -15 \]
\[ -30 - 3y = -15 \]
\[ -3y = 15 \]
\[ y = -5 \]

Solution: \((-6, -5)\)

3. To start this one we need to get the second equation in standard form. The resulting system will be:

\[ 15x - 21y = -63 \]
\[ -5x + 7y = 21 \]

This time we just need to multiply the second equation by 3 to eliminate \( x \):

\[ 15x - 21y = -63 \Rightarrow +15x - 21y = -63 \]
\[ 3(-5x + 7y = 21) \Rightarrow -15x + 21y = -63 \]
\[ \begin{align*}
0y &= 0 \\
y &= 0
\end{align*} \]

Solution: \((0, 0)\)
Solution: There are infinite solutions.

Problem Set
Solve the systems using linear combinations.

1. 
   
   \[ 17x - 5y = 4 \]
   \[ 2x + 7y = 46 \]

2. 
   
   \[ 9x + 2y = -13 \]
   \[ 11x + 5y = 2 \]

3. 
   
   \[ 3x + 4y = -16 \]
   \[ 5x + 5y = -5 \]

4. 
   
   \[ 2x - 8y = -8 \]
   \[ 3x + 7y = 45 \]

5. 
   
   \[ 5x - 10y = 60 \]
   \[ 6x + 3y = -33 \]

6. 
   
   \[ 3x + 10y = -50 \]
   \[ -5x - 7y = 6 \]

7. 
   
   \[ 11x + 6y = 30 \]
   \[ 13x - 5y = -25 \]

8. 
   
   \[ 15x + 2y = 23 \]
   \[ 18x - 9y = -18 \]

9. 
   
   \[ 12x + 8y = 64 \]
   \[ 17x - 12y = 9 \]
10. 

\[11x - 3y = 12\]
\[33x - 36 = 9y\]

11. 

\[4x + 3y = 0\]
\[6x - 13y = 35\]

12. 

\[18x + 2y = -2\]
\[-12x - 3y = -7\]

13. 

\[-6x + 11y = -109\]
\[8x - 15y = 149\]

14. 

\[8x = -5y - 1\]
\[-32x + 20y = 8\]

15. 

\[10x - 16y = -12\]
\[-15x + 14y = -27\]

Set up and solve a system of equations to answer the following questions.

16. A mix of 35% almonds and 65% peanuts sells for $5.70. A mix of 75% almonds and 25% peanuts sells for $6.50. How much should a mix of 60% almonds and 40% peanuts sell for?

17. The Robinson family pays $19.75 at the movie theater for 3 medium popcorns and 4 medium drinks. The Jamison family pays $33.50 at the same theater for 5 medium popcorns and 7 medium drinks. How much would it cost for a couple to get 2 medium drinks and 2 medium popcorns?

18. A cell phone company charges extra when users exceed their included call time and text message limits. One user paid $3.24 extra having talked for 240 extra minutes and sending 12 additional texts. A second user talked for 120 extra minutes and sent 150 additional texts and was charged $4.50 above the regular fee. How much extra would a user be charged for talking 140 extra minutes and sending 200 additional texts?

---

**Determining the Best Method (Graphing, Substitution, Linear Combinations)**

**Objective**

Solve systems of linear equations using the most efficient method.

**Guidance**

Any of the methods (graphing, substitution, linear combination) learned in this unit can be used to solve a linear system of equations. Sometimes, however it is more efficient to use one method over another based on how the equations are presented. For example
• If both equations are presented in slope intercept form \((y = mx + b)\), then either graphing or substitution would be most efficient.
• If one equation is given in slope intercept form or solved for \(x\), then substitution might be easiest.
• If both equations are given in standard form \((Ax + By = C)\), then linear combinations is usually most efficient.

**Example A**

Solve the following system:

\[
\begin{align*}
  y &= -x + 5 \\
  y &= \frac{1}{2}x + 2
\end{align*}
\]

**Solution:** Since both equations are in slope intercept form we could easily graph these lines. The question is whether or not the intersection of the two lines will lie on the “grid” (whole numbers). If not, it is very difficult to determine an answer from a graph. One way to get around this difficulty is to use technology to graph the lines and find their intersection.

The first equation has a \(y\)–intercept of 5 and slope of -1. It is shown here graphed in **blue**.

The second equation has a \(y\)–intercept of 2 and a slope of \(\frac{1}{2}\). It is shown here graphed in **red**.

The two lines clearly intersect at (2, 3).

Alternate Method: Substitution may be the preferred method for students who would rather solve equations algebraically. Since both of these equations are equal to \(y\), we can let the right hand sides be equal to each other and solve for \(x\):
1.3. Solving Linear Systems by Linear Combinations (Elimination)

\[-x + 5 = \frac{1}{2}x + 2\]
\[2\left(-x + 5 = \frac{1}{2}x + 2\right) \rightarrow \text{Multiplying the equation by 2 eliminates the fraction.}\]
\[-2x + 10 = x + 4\]
\[6 = 3x\]
\[x = 2\]

Now solve for \(y\):

\[y = -(2) + 5\]
\[y = 3\]

Solution: (2, 3)

**Example B**

Solve the system:

\[15x + y = 24\]
\[y = -4x + 2\]

**Solution:** This time one of our equations is already solved for \(y\). It is easiest here to use this expression to substitute into the other equation and solve:

\[15x + (-4x + 2) = 24\]
\[15x - 4x + 2 = 24\]
\[11x = 22\]
\[x = 2\]

Now solve for \(y\):

\[y = -4(2) + 2\]
\[y = -8 + 2\]
\[y = -6\]

Solution: (2, -6)

Check your answer:

\[15(2) + (-6) = 30 - 6 = 24\]
\[-6 = -4(2) + 2 = -8 + 2 = -6 \quad \square\]
Example C
Solve the system:

\[-6x + 11y = 86\]
\[9x - 13y = -115\]

Solution: Both equations in this example are in standard form so the easiest method to use here is linear combinations. Since the LCM of 6 and 9 is 18, we will multiply the first equation by 3 and the second equation by 2 to eliminate \(x\) first:

\[3(-6x + 11y = 86) \Rightarrow -18x + 33y = 258\]
\[2(9x - 13y = -115) \Rightarrow 18x - 26y = -230\]
\[7y = 28\]
\[y = 4\]

Now solve for \(x\):

\[-6x + 11(4) = 86\]
\[-6x + 44 = 86\]
\[-6x = 42\]
\[x = -7\]

Solution: (-7, 4)
Check your answer:

\[-6(-7) + 11(4) = 42 + 44 = 86\]
\[9(-7) - 13(4) = -63 - 52 = -115\]

Example D
A rental car company, Affordable Autos, charges $30 per day plus $0.51 per mile driven. A second car rental company, Cheap Cars, charges $25 per day plus $0.57 per mile driven. For a short distance, Cheap Cars offers the better deal. At what point (after how many miles in a single day) does the Affordable Autos rental company offer the better deal? Set up and solve a system of linear equations using technology.

Solution: First set up equations to represent the total cost (for one day’s rental) for each company:

Affordable Autos \(\Rightarrow y = 0.51x + 30\)
Cheap cars \(\Rightarrow y = 0.57x + 25\)

Now graph and solve this system using technology.

You may need to play with the graph window on the calculator to be able to view the intersection. A good window is: \(0 \leq x \leq 125\) and \(0 \leq y \leq 125\). Once we have the intersection in the viewing window we can go to the Calc menu and select Intersect. Now, select each of the lines and press enter to find the intersection: (83.3, 72.5). So, Affordable Autos has a better deal if we want to drive more than 83.3 miles during our one day rental.
Guided Practice
Solve the following systems using the most efficient method:
1.
\[ y = -3x + 2 \]
\[ y = 2x - 3 \]

2.
\[ 4x + 5y = -5 \]
\[ x = 2y - 11 \]

3.
\[ 4x - 5y = -24 \]
\[ -15x + 7y = -4 \]

Answers
1. This one could be solved by graphing, graphing with technology or substitution. This time we will use substitution. Since both equations are solved for \( y \), we can set them equal and solve for \( x \):

\[ -3x + 2 = 2x - 3 \]
\[ 5 = 5x \]
\[ x = 1 \]

Now solve for \( y \):

\[ y = -3(x) + 2 \]
\[ y = -3 + 2 \]
\[ y = -1 \]

Solution: \((1, -1)\)

2. Since the second equation here is solved for \( x \), it makes sense to use substitution:

\[ 4(2y - 11) + 5y = -5 \]
\[ 8y - 44 + 5y = -5 \]
\[ 13y = 39 \]
\[ y = 3 \]

Now solve for \( x \):

\[ x = 2(3) - 11 \]
\[ x = 6 - 11 \]
\[ x = -5 \]
Solution: (-5, 3)

3. This time, both equations are in standard form so it makes the most sense to use linear combinations. We can eliminate \( y \) by multiplying the first equation by 7 and the second equation by 5:

\[
\begin{align*}
7(4x - 5y = -24) & \Rightarrow 28x - 35y = -168 \\
5(-15x + 7y = -4) & \Rightarrow -75x + 35y = -20
\end{align*}
\]

\[
\begin{array}{c}
28x - 35y = -168 \\
-75x + 35y = -20
\end{array}
\]

\[
\begin{align*}
-47x & = -188 \\
x & = 4
\end{align*}
\]

Now find \( y \):

\[
\begin{align*}
4(4) - 5y & = -24 \\
16 - 5y & = -24 \\
-5y & = -40 \\
y & = 8
\end{align*}
\]

Solution: (4, 8)

**Problem Set**

Solve the following systems using linear combinations.

1. 
\[
\begin{align*}
5x - 2y & = -1 \\
8x + 4y & = 56
\end{align*}
\]

2. 
\[
\begin{align*}
3x + y & = -16 \\
-4x - y & = 21
\end{align*}
\]

3. 
\[
\begin{align*}
7x + 2y & = 4 \\
y & = -4x + 1
\end{align*}
\]

4. 
\[
\begin{align*}
6x + 5y & = 25 \\
x & = 2y + 24
\end{align*}
\]

5. 
\[
\begin{align*}
-8x + 10y & = -1 \\
2x - 6y & = 2
\end{align*}
\]
6. 
\[\begin{align*}
3x + y &= 18 \\
-7x + 3y &= -10
\end{align*}\]

7. 
\[\begin{align*}
2x + 15y &= -3 \\
-3x - 5y &= -6
\end{align*}\]

8. 
\[\begin{align*}
15x - y &= 19 \\
13x + 2y &= 48
\end{align*}\]

9. 
\[\begin{align*}
x &= -9y - 2 \\
-2x - 15y &= 6
\end{align*}\]

10. 
\[\begin{align*}
3x - 4y &= 1 \\
-2x + 3y &= 1
\end{align*}\]

11. 
\[\begin{align*}
x - y &= 2 \\
3x - 2y &= -7
\end{align*}\]

12. 
\[\begin{align*}
3x + 12y &= -18 \\
y &= -\frac{1}{4}x - \frac{3}{2}
\end{align*}\]

13. 
\[\begin{align*}
-2x - 8y &= -2 \\
x &= \frac{1}{2}y + 10
\end{align*}\]

14. 
\[\begin{align*}
14x + y &= 3 \\
-21x - 3y &= -3
\end{align*}\]

15. 
\[\begin{align*}
y &= \frac{4}{5}x + 7 \\
8x - 10y &= 2
\end{align*}\]
Solve the following word problem by creating and solving a system of linear equations.

16. Jack and James each buy some small fish for their new aquariums. Jack buys 10 clownfish and 7 goldfish for $28.25. James buys 5 clownfish and 6 goldfish for $17.25. How much does each type of fish cost?

17. The sum of two numbers is 35. The larger number is one less than three times the smaller number. What are the two numbers?

18. Rachel offers to go to the coffee shop to buy cappuccinos and lattes for her coworkers. She buys a total of nine drinks for $35.75. If cappuccinos cost $3.75 each and the lattes cost $4.25 each, how many of each drink did she buy?
1.4 Graphing and Solving Linear Inequalities

Objective
Identify solutions to and solve linear inequalities.

Review Queue
1. Solve the inequalities:
   a) \(8 - 3x \leq 7\)
   b) \(\frac{3}{4}x > -9\)
   c) \(|5x - 2| < 8\)
2. Graph the linear inequality: \(y \geq 3x - 5\)
3. Graph the linear inequality: \(2x + 5y < 10\)

Checking for Solutions to a System of Linear Inequalities

Objective
Determine whether or not a given point is a solution to a linear system of inequalities.

Watch This

Khan Academy: Testing Solutions for a System of Inequalities

Guidance
A linear system of inequalities has an infinite number of solutions. Recall that when graphing a linear inequality the solution is a shaded region of the graph which contains all the possible solutions to the inequality. In a system, there are two linear inequalities. The solution to the system is all the points that satisfy both inequalities or the region in which the shading overlaps.

Example A
Given the system of linear inequalities shown in the graph, determine which points are solutions to the system.
Example A
Determine whether the following points are solutions to the system of linear inequalities:

a) \((0, -1)\)
b) \((2, 3)\)
c) \((-2, -1)\)
d) \((3, 5)\)

**Solution:**
a) The point \((0, -1)\) is not a solution to the system of linear inequalities. It is a solution to \(y \leq \frac{2}{3}x + 3\) (graphed in blue), but it lies on the line \(y = -\frac{4}{5}x - 1\) which is not included in the solution to \(y > -\frac{4}{5}x - 1\) (shown in red). The point must satisfy both inequalities to be a solution to the system.

b) The point \((2, 3)\) lies in the overlapping shaded region and therefore is a solution to the system.

c) The point \((-2, -1)\) lies outside the overlapping shaded region and therefore is not a solution to the system.

d) The point \((3, 5)\) lies on the line \(y = \frac{2}{3}x + 3\), which is included in the solution to \(y \leq \frac{2}{3}x + 3\). Since this part of the line is included in the solution to \(y > -\frac{4}{5}x - 1\), it is a solution to the system.

**Example B**
Determine whether the following points are solutions to the system of linear inequalities:

\[
\begin{align*}
3x + 2y & \geq 4 \\
x + 5y & < 11
\end{align*}
\]

a) \((3, 1)\)
b) \((1, 2)\)
c) \((5, 2)\)
d) \((-3, 1)\)

**Solution:** This time we do not have a graph with which to work. Instead, we will plug the points into the equations to determine whether or not they satisfy the linear inequalities. A point must satisfy both linear inequalities to be a solution to the system.
1.4. Graphing and Solving Linear Inequalities

a) Yes, $3(3) + 2(1) \geq 4$ ✓ and $(3) + 5(1) < 11$ ✓. Therefore, $(3, 1)$ is a solution to the system.
b) No, $3(1) + 2(2) \geq 4$ ✓, but $(1) + 5(2) = 11$, so the point fails the second inequality.
c) No, $3(5) + 2(2) \geq 4$ ✓, but $(5) + 5(2) > 11$, so the point fails the second inequality.
d) No, $3(-3) + 2(1) = -9 + 2 = -7 < 4$, so the point fails the first inequality. There is no need to check the point in the second inequality since it must satisfy both to be a solution.

Guided Practice

1. Determine whether the given points are solutions to the systems shown in the graph:
   a) (-3, 3)
   b) (4, 2)
   c) (3, 2)
   d) (-4, 4)

2. Determine whether the following points are solutions to the system:
   \[ y < 11x - 5 \]
   \[ 7x - 4y \geq 1 \]
   a) (4, 0)
   b) (0, -5)
   c) (7, 12)
   d) (-1, -3)

Answers

1. a) (-3, 3) is a solution to the system because it lies in the overlapping shaded region.
b) (4, 2) is not a solution to the system. It is a solution to the red inequality only.

c) (3, 2) is not a solution to the system because it lies on the dashed blue line and therefore does not satisfy that inequality.

d) (−4, 4) is a solution to the system since it lies on the solid red line that borders the overlapping shaded region.

2. a) Yes, \(0 < 11(4) - 5\) and \(7(4) - 4(0) \geq 1\)

b) No, \(−5 = 11(0) - 5\) so the first inequality is not satisfied.

c) Yes, \(12 < 11(7) - 5\) and \(7(7) - 4(12) \geq 1\)

d) No, \(−3 > 11(−1) - 5\) so the first inequality is not satisfied.

**Problem Set**

Given the four linear systems graphed below, match the point with the system(s) for which it is a solution.

A.

B.
C.

D.
Given the four linear systems below, match the point with the system(s) for which it is a solution.

A. \[
\begin{align*}
5x + 2y &\leq 10 \\
3x - 4y &> -12
\end{align*}
\]

B. \[
\begin{align*}
5x + 2y &< 10 \\
3x - 4y &\leq -12
\end{align*}
\]

C. \[
\begin{align*}
5x + 2y &> 10 \\
3x - 4y &< -12
\end{align*}
\]

D. \[
\begin{align*}
5x + 2y &\geq 10 \\
3x - 4y &\geq -12
\end{align*}
\]
11. (0, 0)  
12. (4, 6)  
13. (0, 5)  
14. (-3, 4)  
15. (4, 3)  
16. (0, 3)  
17. (-8, -3)  
18. (1, 6)  
19. (4, -5)  
20. (4, -2)

---

### Graphing Systems of Linear Inequalities

**Objective**

Graph linear systems of inequalities with two or three equations and identify the region representing the solution set.

**Watch This**

[James Sousa: Ex 1: Graph a System of Linear Inequalities](http://www.ck12.org/flx/render/embeddedobject/60098)

**Guidance**

In this section we will be graphing two and three linear inequalities on the same grid and identifying where the shaded regions overlap. This overlapping region is the solution to the system. Note: If the shaded regions do not overlap, there is no solution as shown in Example B.

**Example A**

Graph and identify the solution to the system:

\[ y > 2x - 3 \]

\[ y \leq 4x + 1 \]

**Solution:**

Since both of these inequalities are given in slope intercept form, we can use the \( y \)-intercept and the slope to graph the lines. Since inequality 1 has “\( y > \)”, we will make a dashed line to indicate that the line is not included in the solution and shade above the line where \( y \) is “greater” (where the \( y \)-axis is above) the line. Since inequality 2 has “\( y \leq \)” we will make a solid line to indicate that the line is included in the solution set and shade below the line where \( y \) is “less than” (where the \( y \)-axis is below) the line. Inequality 1 is graphed in blue and inequality 2 is graphed in red. The overlap of the shaded regions (purple shading) represents the solution.
Example B

Graph and identify the solution to the system:

\[
y \geq -\frac{2}{3}x + 2 \\
y \leq -\frac{2}{3}x - 5
\]

Solution:

Since inequality 1 has “\(y \geq\)”, we will make a solid line to indicate that the line is included in the solution and shade above the line where \(y\) is “greater” (where the \(y\)-axis is above) the line. Since inequality 2 has “\(y \leq\)” we will make a dashed line to indicate that the line is not included in the solution set and shade below the line where \(y\) is “less than” (where the \(y\)-axis is below) the line. Inequality 1 is graphed in blue and inequality 2 is graphed in red. In this case the regions do not overlap. This indicates that there is no solution to the system.
Example C

Graph and identify the solution to the system:

\[
3x - y < 6 \\
8x + 5y \leq 40
\]

Solution: This time, let’s use a different graphing technique. We can identify the intercepts for each equation and graph the lines using these points:

For \(3x - y < 6\), the intercepts are \((2, 0)\) and \((0, -6)\).

For \(8x + 5y \leq 40\), the intercepts are \((5, 0)\) and \((0, 8)\).
For the first inequality, the symbol is < so the line is dashed. Now, use a test point to determine which way to shade. (0, 0) is an easy point to test. \(3(0) - (0) < 6\) Since this is a true statement, (0, 0) is a solution to the inequality and we can shade on the side of the line with (0, 0).

For the second inequality, the symbol is \(\leq\) so the line is solid. Using the same test point, \((0, 0)\), \(8(0) + 5(0) \leq 40\) This is a true statement so (0, 0) is a solution to the inequality and we can shade on the side of the line with (0, 0).

Again, Inequality 1 is graphed in blue and inequality 2 is graphed in red. The overlap of the shaded regions (purple shading) represents the solution.

**Example D**

Graph the system of linear inequalities:

\[
y < -\frac{2}{3}x + 3 \\
y \geq 1 \\
x \geq -4
\]

**Solution:**

As in the previous concept, we will graph the lines and determine whether each line should be dashed or solid and which way to shade.

\(y < -\frac{2}{3}x + 3\) This inequality has a y-intercept of 3 and slope of \(-\frac{2}{3}\). Since the inequality is <, we will shade below the dashed blue line.

\(y \geq 1\) This is a horizontal line through (0, 1). The line will be solid and we shade above the red line.

\(x \geq -4\) This is a vertical line through (-4, 0). The line will be solid and we will shade (yellow) to the right of the green line.

The solution to this system is the shaded region (triangular) in the center where all three shaded regions overlap. This region can be difficult to see in a graph so it is common practice to erase the shading that is not a part of the solution to make the solution region is more obvious.
Guided Practice

Graph and identify the solutions to the systems.

1. 
\[
\begin{align*}
y & \leq \frac{1}{3}x + 5 \\
y & > \frac{5}{4}x - 2
\end{align*}
\]

2. 
\[
\begin{align*}
4x + y & > 8 \\
3x - 5y & \leq 15
\end{align*}
\]

3. 
\[
\begin{align*}
7x + 2y & \leq 14 \\
3x - 9y & \geq 18
\end{align*}
\]

4. 
\[
\begin{align*}
y & \geq 2x - 3 \\
2x + y & > -8 \\
y & > -3
\end{align*}
\]

Answers

In each of the solutions below, the first inequality in the system is shown in blue and the second inequality is shown in red. The solution set is the overlapping shaded region in purple. When there are three inequalities, only the solution region is shown to eliminate confusion.

1.
The inequalities in this system are both already in slope intercept form so we can graph them using the slope and $y-$intercept of each line and shade as shown below.

$y \leq \frac{1}{3}x + 5 \Rightarrow$ solid line and shade below

$y > \frac{5}{4}x - 2 \Rightarrow$ dashed line and shade above

In these inequalities it is easiest to graph using the $x$ and $y$ intercepts. Once we have graphed the lines we can use a test point to determine which side should be shaded.

$4x + y > 8 \Rightarrow$ The intercepts are $(2, 0)$ and $(0, 8)$ and the line will be dashed. If we test the point $(0, 0)$, the inequality is not true so we shade on the side of the line that does not contain $(0, 0)$. 
3x - 5y ≤ 15 ⇒ The intercepts are (5, 0) and (0, -3) and the line will be solid. The test point (0, 0) satisfies the inequality so we shade on the side of the line that includes (0, 0).

3.

Again, it is easiest here to graph using the x and y intercepts. Once we have graphed the lines we can use a test point to determine which side should be shaded.

7x + 2y ≤ 14 ⇒ The intercepts are (2, 0) and (0, 7) and the line will be solid. If we test the point (0, 0), the inequality is true so we shade on the side of the line that contains (0, 0).

3x - 9y ≥ 18 ⇒ The intercepts are (6, 0) and (0, -2) and the line will be solid. The test point (0, 0) does not satisfy the inequality so we shade on the side of the line that does not include (0, 0).

4.
Inequality 1 can be graphed using the slope and \( y \)-intercept. This line will be solid and the shading will be above this line.

Inequality 2 can be graphed using intercepts. The line will be dashed and we can use a test point to determine that the shaded region will be above this line.

Inequality 3 is a horizontal line. It will be dashed and the shading is above this line.

The intersection of these three regions is shaded in purple on the graph.

**Problem Set**

Graph the following systems of linear inequalities.

1. \[ y > \frac{1}{2}x - 2 \]
   \[ 4x + 6y \leq 24 \]

2. \[ y > -\frac{3}{4}x - 1 \]
   \[ y > 3x + 5 \]

3. \[ y \leq -\frac{2}{3}x + 2 \]
   \[ y \geq -\frac{5}{3}x - 1 \]

4. \[ y \geq -x - 3 \]
   \[ y < \frac{1}{5}x + 1 \]

5. \[ 5x - 2y > -10 \]
   \[ y \leq -\frac{1}{3}x + 2 \]

6. \[ y > -\frac{4}{5}x - 3 \]
   \[ y > x \]

7. \[ y \leq \frac{1}{2}x + 4 \]
   \[ x - 2y \leq 2 \]
8. \[7x - 3y > -21\]
   \[x - 4y < 8\]

9. \[6x + 5y \leq 5\]
   \[2x - 3y \leq 12\]

10. \[x < 3\]
    \[y \geq 2x + 1\]

11. \[y < 2\]
    \[y \geq -2\]

12. \[2x - y \leq 4\]
    \[5x + 2y > 10\]

13. \[y \leq -2x + 4\]
    \[y \geq 5x + 4\]
    \[y > \frac{1}{2}x - 1\]

14. \[x + y \leq 3\]
    \[x \leq 3\]
    \[y < 3\]

15. \[x > -2\]
    \[y > -3\]
    \[2x + y \leq 2\]

16. \[x > -2\]
    \[x \leq 4\]
    \[3x + 5y > 15\]
17.

\begin{align*}
2x + 3y & > 6 \\
5x - 2y & < -10 \\
x - 3y & > 3
\end{align*}

18.

\begin{align*}
y & \leq x \\
y & \geq -x \\
x & < 5
\end{align*}
1.5 Solving Linear Systems in Three Variables

Objective
Identify solutions to and solve linear systems in three variables.

Review Queue
1. Is the point (-6, 4) the solution to the system:

\[
\begin{align*}
2x + 3y &= 0 \\
x + y &= -3
\end{align*}
\]

2. Solve the system using linear combinations:

\[
\begin{align*}
2x + 8y &= -6 \\
x - y &= 7
\end{align*}
\]

3. Describe the situation (geometrically) for a linear system with no solution.

Solving a System in Three Variables Using Linear Combinations

Objective
Understand the geometric situations that occur when there is one solution, infinite solutions and no solution to a system of equations in three variables. Solve systems in three variables and verify solutions to the system.

Guidance
An equation in three variables, such as \(2x - 3y + 4z = 10\), is an equation of a plane in three dimensions. In other words, this equation expresses the relationship between the three coordinates of each point on a plane. The solution to a system of three equations in three variables is a point in space which satisfies all three equations. When we add a third dimension we use the variable, \(z\), for the third coordinate. For example, the point (3, -2, 5) would be \(x = 3, y = -2\) and \(z = 5\). A solution can be verified by substituting the \(x, y,\) and \(z\) values into the equations to see if they are valid.

A system of three equations in three variables consists of three planes in space. These planes could intersect with each other or not as shown in the diagrams below.

- a unique solution exists
- no solution
- infinite solutions
• In the first diagram, the three planes intersect at a single point and thus a unique solution exists and can be found.
• The second diagram illustrates one way that three planes can exist and there is no solution to the system. It is also possible to have three parallel planes or two that are parallel and a third that intersects them. In any of these cases, there is no point that is in all three planes.
• The third diagram shows three planes intersecting in a line. Every point on this line is a solution to the system and thus there are infinite solutions.

To solve a system of three equations in three variables, we will be using the linear combination method. This time we will take two equations at a time to eliminate one variable and using the resulting equations in two variables to eliminate a second variable and solve for the third. This is just an extension of the linear combination procedure used to solve systems with two equations in two variables.

Example A
Determine whether the point, (6, -2, 5), is a solution to the system:

\[
\begin{align*}
    x - y + z &= 13 \\
    2x + 5y - 3z &= -13 \\
    4x - y - 6z &= -4
\end{align*}
\]

Solution: In order for the point to be a solution to the system, it must satisfy each of the three equations.
First equation: \((6) - (-2) + (5) = 6 + 2 + 5 = 13 \, \checkmark\)
Second equation: \(2(6) + 5(-2) - 3(5) = 12 - 10 - 15 = -13 \, \checkmark\)
Third equation: \(4(6) - (-2) - 6(5) = 24 + 2 - 30 = -4 \, \checkmark\)
The point, \((6, -2, 5)\), satisfies all three equations. Therefore, it is a solution to the system.

Example B
Solve the system using linear combinations:

\[
\begin{align*}
    2x + 4y - 3z &= -7 \\
    3x - y + z &= 20 \\
    x + 2y - z &= -2
\end{align*}
\]

Solution: We can start by taking two equations at a time and eliminating the same variable. We can take the first two equations and eliminate \(z\), then take the second and third equations and also eliminate \(z\).

\[
\begin{align*}
    2x + 4y - 3z &= -7 \\
    3(3x - y + z) &= 20 \\
    9x - 3y + 3z &= 60 \\
    11x + y &= 53
\end{align*}
\]

Result from equations 1 and 2: \(11x + y = 53\)
Result from equations 2 and 3: $4x + y = 18$

Now we have reduced our system to two equations in two variables. We can eliminate $y$ most easily next and solve for $x$.

\[
\begin{align*}
11x + y &= 53 \\
-1(4x + y) &= 18
\end{align*}
\Rightarrow
\begin{align*}
11x + y &= 53 \\
-4x - y &= -18
\end{align*}
\Rightarrow
\begin{align*}
7x &= 35 \\
x &= 5
\end{align*}
\]

Now use this value to find $y$:

\[
\begin{align*}
4(5) + y &= 18 \\
20 + y &= 18 \\
y &= -2
\end{align*}
\]

Finally, we can go back to one of the original three equations and use our $x$ and $y$ values to find $z$.

\[
\begin{align*}
2(5) + 4(-2) - 3z &= -7 \\
10 - 8 - 3z &= -7 \\
2 - 3z &= -7 \\
-3z &= -9 \\
z &= 3
\end{align*}
\]

Therefore the solution is $(5, -2, 3)$.

Don’t forget to check your answer by substituting the point into each equation.

Equation 1: $2(5) + 4(-2) - 3(3) = 10 - 8 - 9 = -7 \, \checkmark$

Equation 2: $3(5) - (-2) + (3) = 15 + 2 + 3 = 20 \, \checkmark$

Equation 3: $(5) + 2(-2) - (3) = 5 - 4 - 3 = -2 \, \checkmark$

**Example C**

Solve the system using linear combinations:

\[
\begin{align*}
x - 3y + 4z &= 14 \\
-x + 2y - 5z &= -13 \\
2x + 5y - 3z &= -5
\end{align*}
\]

**Solution:** In this case, it is easiest to eliminate $x$ first by combining the first two equations and then combining the second and third equations.

\[
\begin{align*}
x - 3y + 4z &= 14 \\
-x + 2y - 5z &= -13
\end{align*}
\Rightarrow
\begin{align*}
x - 3y + 4z &= 14 \\
-1x + 2y - 5z &= -13 \\
y - z &= 1
\end{align*}
\]
Result from equations 1 and 2: \(-y - z = 1\)

\[
2(-x + 2y - 5z = -13) \Rightarrow -2x + 4y - 10z = -26
\]
\[
2x + 5y - 3z = -5
\]
\[
\frac{2x + 5y - 3z = -5}{9y - 13z = -31}
\]

Result from equations 2 and 3: \(9y - 13z = -31\)

Now we have reduced our system to two equations in two variables. We can eliminate \(y\) most easily next and solve for \(z\).

\[
9(-y - z = 1) \Rightarrow -9y - 9z = 9
\]
\[
9y - 13z = -31
\]
\[
\frac{9y - 13z = -31}{-22z = -22}
\]
\[
z = 1
\]

Now use this value to find \(y\):

\[
-y - (1) = 1
\]
\[
-y - 1 = 1
\]
\[
-y = 2
\]
\[
y = -2
\]

Finally, we can go back to one of the original three equations and use our \(y\) and \(z\) values to find \(x\).

\[
x - 3(-2) + 4(1) = 14
\]
\[
x + 6 + 4 = 14
\]
\[
x + 10 = 14
\]
\[
x = 4
\]

Therefore the solution is (4, -2, 1).

Don’t forget to check your answer by substituting the point into each equation.

Equation 1: \((4) - 3(-2) + 4(1) = 4 + 6 + 4 = 14 \checkmark\)

Equation 2: \(- (4) + 2(-2) - 5(1) = -4 - 4 - 5 = -13 \checkmark\)

Equation 3: \(2(4) + 5(-2) - 3(1) = 8 - 10 - 3 = -5 \checkmark\)

**Example D**

Solve the system using linear combinations:

\[
x + y + z = 5
\]
\[
5x + 5y + 5z = 20
\]
\[
2x + 3y - z = 8
\]
Solution: We can start by combining equations 1 and 2 together by multiplying the first equation by -5.

\[-5(x + y + z = 5) \Rightarrow -5x - 5y - 5z = -25\]
\[5x + 5y + 5z = 20\]
\[0 = -5\]

Since the result is a false equation, there is no solution to the system.

Example E
Solve the system using linear combinations:

\[x + y + z = 3\]
\[x + y - z = 3\]
\[2x + 2y + z = 6\]

Solution: We can start by combining the first two equations and then combine equations 2 and 3.

\[x + y + z = 3\]
\[x + y - z = 3\]
\[2x + 2y = 6\]

Result from equations 1 and 2: \[2x + 2y = 6\]

\[x + y - z = 3\]
\[2x + 2y + z = 6\]
\[3x + 3y = 9\]

Result from equations 2 and 3: \[3x + 3y = 9\]

Now we can combine these two equations and attempt to eliminate \(x\) or \(y\).

\[3(2x + 2y = 6) \Rightarrow 6x + 6y = 18\]
\[-2(3x + 3y = 9) \Rightarrow -6x - 6y = -18\]
\[0 = 0\]

This is a true statement. Therefore, there are infinite solutions to this system.

Guided Practice
1. Is the point, (-3, 2, 1), a solution to the system:

\[x + y + z = 0\]
\[4x + 5y + z = -1?\]
\[3x + 2y - 4z = -8\]
2. Solve the following system using linear combinations:

\[
\begin{align*}
5x - 3y + z &= -1 \\
x + 6y - 4z &= -17 \\
8x - y + 5z &= 12
\end{align*}
\]

3. Solve the following system using linear combinations:

\[
\begin{align*}
2x + y - z &= 3 \\
x - 2y + z &= 5 \\
6x + 3y - 3z &= 6
\end{align*}
\]

**Answers**

1. Check to see if the point satisfies all three equations.

   - **Equation 1:** 
     \[(-3) + (2) + (1) = -3 + 2 + 1 = 0 \checkmark\]
   
   - **Equation 2:** 
     \[4(-3) + 5(2) + (1) = -12 + 10 + 1 = -1 \checkmark\]
   
   - **Equation 3:** 
     \[3(-3) + 2(2) - 4(1) = -9 + 4 - 4 = -9 \neq -8 \checkmark\]

   Since the third equation is not satisfied by the point, the point is not a solution to the system.

2. Combine the first and second equations to eliminate \(z\). Then combine the first and third equations to eliminate \(z\).

   \[
   \begin{align*}
   4(5x - 3y + z &= -1) &\Rightarrow & 20x - 12y + 4z &= -4 \\
x + 6y - 4z &= -17 &\Rightarrow & x + 6y - 4z &= -17 \\
&\Rightarrow & 21x - 6y &= -21
   \end{align*}
   \]

   Result from equations 1 and 2: \(21x - 6y = -21\)

   \[
   \begin{align*}
   -5(5x - 3y + z &= -1) &\Rightarrow & -25x + 15y - 5z &= 5 \\
8x - y + 5z &= 12 &\Rightarrow & 8x - y + 5z &= 12 \\
&\Rightarrow & -17x + 14y &= 17
   \end{align*}
   \]

   Result from equations 1 and 3: \(-17x + 14y = 17\)

   Now we have reduced our system to two equations in two variables. We can eliminate \(y\) most easily next and solve for \(x\).

   \[
   \begin{align*}
   7(21x - 6y &= -21) &\Rightarrow & 147x - 42y &= -147 \\
3(-17x + 14y &= 17) &\Rightarrow & -51x + 42y &= 51
   \end{align*}
   \]

   \[
   \begin{align*}
   &\Rightarrow & 96x &= -96 \\
x &= -1
   \end{align*}
   \]
Now find $y$:

\[
21(-1) - 6y = -21 \\
-21 - 6y = -21 \\
-6y = 0 \\
y = 0
\]

Finally, we can go back to one of the original three equations and use our $y$ and $z$ values to find $x$.

\[
5(-1) - 3(0) + z = -1 \\
-5 + z = -1 \\
z = 4
\]

Therefore the solution is $(-1, 0, 4)$.

Don’t forget to check your answer by substituting the point into each equation.

**Equation 1:** $5(-1) - 3(0) + (4) = -5 + 4 = -1 \checkmark$

**Equation 2:** $(-1) + 6(0) - 4(4) = -1 - 16 = -17 \checkmark$

**Equation 3:** $8(-1) - (0) + 5(4) = -8 + 20 = 12 \checkmark$

3. Combine equations 1 and two to eliminate $z$. Then combine equations 2 and 3 to eliminate $z$.

\[
\begin{align*}
2x + y - z &= 3 \\
x - 2y + z &= 5 \\
3x - y &= 8
\end{align*}
\]

Result from equations 1 and 2: $3x - y = 8$

\[
\begin{align*}
3(x - 2y + z = 5) &\Rightarrow 3x - 6y + 3z = 15 \\
6x + 3y - 3z &= 6 & 6x + 3y - 3z &= 6 \\
9x - 3y &= 21
\end{align*}
\]

Result from equations 2 and 3: $9x - 3y = 21$

Now we have reduced our system to two equations in two variables. Now we can combine these two equations and attempt to eliminate another variable.

\[
\begin{align*}
-3(3x - y = 8) &\Rightarrow -9x + 3y = -24 \\
9x - 3y &= 21 & 9x - 2y &= 21 \\
0 &= -3
\end{align*}
\]

Since the result is a false equation, there is no solution to the system.

**Problem Set**
1. Is the point, (2, -3, 5), the solution to the system:

\[
\begin{align*}
2x + 5y - z &= -16 \\
5x - y - 3z &= -2 \\
3x + 2y + 4z &= 20
\end{align*}
\]

2. Is the point, (-1, 3, 8), the solution to the system:

\[
\begin{align*}
8x + 10y - z &= 14 \\
11x + 4y - 3z &= -23 \\
2x + 3y + z &= 10
\end{align*}
\]

3. Is the point, (0, 3, 5), the solution to the system:

\[
\begin{align*}
5x - 3y + 2z &= 1 \\
7x + 2y - z &= 1 \\
x + 4y - 3z &= -3
\end{align*}
\]

Solve the following systems in three variables using linear combinations.

4.

\[
\begin{align*}
3x - 2y + z &= 0 \\
4x + y - 3z &= -9 \\
9x - 2y + 2z &= 20
\end{align*}
\]

5.

\[
\begin{align*}
11x + 15y + 5z &= 1 \\
3x + 4y + z &= -2 \\
7x + 13y + 3z &= 3
\end{align*}
\]

6.

\[
\begin{align*}
2x + y + 7z &= 5 \\
3x - 2y - z &= -1 \\
4x - y + 3z &= 5
\end{align*}
\]

7.

\[
\begin{align*}
x + 3y - 4z &= -3 \\
2x + 5y - 3z &= 3 \\
x - 3y + z &= -3
\end{align*}
\]
1.5. Solving Linear Systems in Three Variables

8.

\begin{align*}
3x + 2y - 5z &= -8 \\
3x + 2y + 5z &= -8 \\
6x + 4y - 10z &= -16
\end{align*}

9.

\begin{align*}
x + 2y - z &= -1 \\
2x + 4y + z &= 10 \\
3x - y + 8z &= 6
\end{align*}

10.

\begin{align*}
x + y + z &= -3 \\
2x - y - z &= 6 \\
4x + y + z &= 0
\end{align*}

11.

\begin{align*}
4x + y + 3z &= 8 \\
8x + 2y + 6z &= 15 \\
3x - 3y - z &= 5
\end{align*}

12.

\begin{align*}
2x + 3y - z &= -1 \\
x - 2y + 3z &= -4 \\
-x + y - 2z &= 3
\end{align*}
1.6 References

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