April 3, 2020

To: Families and Caregivers of CMSD Students:

CMSD continues to send regular updates on the services and supports we are providing during the unprecedented closure of schools, as part of a state-wide effort to contain the spread of the COVID-19 virus.

In addition to the grab-and-go meals we are providing at 22 school sites each day, CMSD is also distributing learning packets, and I want to personally emphasize the value of these academic enrichment materials that are handed out with meals and posted on the CMSD website: ClevelandMetroSchools.org.

Research shows that children learn best when learning is continuous, which is why CMSD educators are working hard to produce interesting and thought-provoking materials that will keep students engaged and that will keep their minds active during this long break from school.

Recognizing that students are used to a consistent school schedule, I strongly encourage you to work with your child to develop a routine at home, to make time and space for quiet reading and active engagement with their learning materials and to praise them for their attention to their studies and their personal growth.

CMSD’s Academic Enrichment Plan, posted on CMSD’s website, includes lessons and a recommended daily schedule for students at every grade level, from PreK to 12. Digital lessons can be accessed online and print materials are available for pickup at all meal sites.

Thank you for the opportunity to emphasize the importance of academic enrichment in our students’ experience during this unprecedented time away from school. And thank you for the important role you play every day in our shared commitment to the safety, growth and future of Cleveland’s children.

Thank you.

Eric S. Gordon
CEO
<table>
<thead>
<tr>
<th>Math (40 Minutes)</th>
<th>April 6</th>
<th>April 7</th>
<th>April 8</th>
<th>April 9</th>
<th>April 10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Algebra II CK12 Flexbook: Systems of Linear Equations and Inequalities 1.1</td>
<td>Algebra II CK12 Flexbook: Systems of Linear Equations and Inequalities 1.1</td>
<td>Algebra II CK12 Flexbook: Systems of Linear Equations and Inequalities 1.1</td>
<td>Algebra II CK12 Flexbook: Systems of Linear Equations and Inequalities 1.1</td>
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<tr>
<td>Online Learning Options</td>
<td>ELA I_II Data Plots Article: Determine meaning of words/phrases as they are used in text. Assignment (April 6 – 8)</td>
<td>ELA I_II Data Plots Article: Determine central idea and provide summary of a text.</td>
<td>ELA I_II Data Plots Article: Analyze how the author develops the text.</td>
<td>ELA I_II Short Story: The Castle in the Woods: Identify key details and answer text dependent questions. Assignment</td>
<td>GOOD FRIDAY – NO ENRICHMENT ACTIVITIES TODAY</td>
</tr>
<tr>
<td>All Grade Levels</td>
<td>ELA III_IV Smartphones Put Your Privacy at Risk: Read and answer questions</td>
<td>ELA III_IV What Adolescents Miss When We Let Them Grow Up in Cyberspace: Read and answer questions</td>
<td>ELA III_IV What Adolescents Miss When We Let Them Grow Up in Cyberspace: Read and answer questions</td>
<td>ELA III_IV</td>
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</tr>
<tr>
<td>Science (40 Minutes)</td>
<td>Physical Science Scientific Method Overview: Presentation and Questions</td>
<td>Physical Science The Methods of Science: Read and answer questions</td>
<td>Physical Science Standards of Measurement: Read and answer questions</td>
<td>Physical Science Communication with Graphs: Read and answer questions</td>
<td>GOOD FRIDAY – NO ENRICHMENT ACTIVITIES TODAY</td>
</tr>
<tr>
<td>Online Learning Options</td>
<td>Biology Nature of Science: Read and answer questions</td>
<td>Biology Methods of Science: Read and answer questions</td>
<td>Biology Standards of Measurement: Read and answer questions</td>
<td>Biology Using Graphs to Understand: Read and answer questions</td>
<td></td>
</tr>
<tr>
<td>Physical Science</td>
<td>Chemistry The Methods of Science: Read and answer questions</td>
<td>Chemistry Standards of Measurement: Read and answer questions</td>
<td>Chemistry Standards of Measurement: Read and answer questions</td>
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<tr>
<td>Biology</td>
<td>Chemistry</td>
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<td>Chemistry</td>
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</tbody>
</table>
## Weekly Enrichment Plan: Week of April 6

**Grade: High School**

### Weekly Enrichment Plan

<table>
<thead>
<tr>
<th>Subject</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scientific Method</strong>&lt;br&gt;Overview: Presentation and questions&lt;br&gt;Env. Sci.&lt;br&gt;Nature of Science: Read and answer questions Presentation&lt;br&gt;Readings&lt;br&gt;Graphs</td>
<td></td>
</tr>
<tr>
<td><strong>Physics</strong>&lt;br&gt;Nature of Science: Read and answer questions&lt;br&gt;Methods of Science: Read and answer questions Reinforcement</td>
<td></td>
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<tr>
<td><strong>Physics</strong>&lt;br&gt;Methods of Science: Read and answer questions Reinforcement</td>
<td></td>
</tr>
<tr>
<td><strong>Physics</strong>&lt;br&gt;Standards of Measurement: Read and answer questions Reinforcement Teaching Visual</td>
<td></td>
</tr>
<tr>
<td><strong>Physics</strong>&lt;br&gt;Communicating with Graphs: Read and answer questions Connecting Math to Physics Enrichment</td>
<td></td>
</tr>
<tr>
<td><strong>World History</strong>&lt;br&gt;WWI: Read and answer questions&lt;br&gt;Holocaust Videos: Watch and answer questions&lt;br&gt;US History&lt;br&gt;WWI: Read and answer questions&lt;br&gt;US Govt.&lt;br&gt;Historical Documents: Read and answer questions&lt;br&gt;US Govt.&lt;br&gt;Political Parties: Read and answer questions</td>
<td></td>
</tr>
<tr>
<td><strong>World History</strong>&lt;br&gt;Dictatorships: Read and answer questions&lt;br&gt;US History&lt;br&gt;America and WWII: Read and answer questions&lt;br&gt;US Govt.&lt;br&gt;Political Parties: Read and answer questions&lt;br&gt;Writing Assignment</td>
<td></td>
</tr>
<tr>
<td><strong>World History</strong>&lt;br&gt;Dictatorships: Read and answer questions&lt;br&gt;US History&lt;br&gt;Manzan Camp: Read and answer questions&lt;br&gt;Photos Japanese in America: Read and answer questions&lt;br&gt;Writing Prompt</td>
<td></td>
</tr>
<tr>
<td><strong>World History</strong>&lt;br&gt;Political Parties Lesson 2: Read and answer questions</td>
<td></td>
</tr>
<tr>
<td><strong>Social Studies</strong> (40 Minutes)&lt;br&gt;Online Learning Options: World History&lt;br&gt;US History&lt;br&gt;US Government</td>
<td></td>
</tr>
<tr>
<td><strong>Math</strong>&lt;br&gt;English&lt;br&gt;Science&lt;br&gt;Social Studies</td>
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<tr>
<td><strong>Math</strong>&lt;br&gt;English&lt;br&gt;Science&lt;br&gt;Social Studies</td>
<td></td>
</tr>
<tr>
<td><strong>GOOD FRIDAY – NO ENRICHMENT ACTIVITIES TODAY</strong></td>
<td></td>
</tr>
</tbody>
</table>

### Student Daily Check-Off

(check off each activity that you completed)

- Math
- English
- Science
- Social Studies
**Weekly Enrichment Plan: Week of April 6**

**Grade: High School**

**Suggested Daily Schedule: Grades 9 - 12**

<table>
<thead>
<tr>
<th>Time</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>8:00 – 9:00 am</td>
<td>Wake up, make your bed, eat breakfast and get ready for an awesome day!</td>
</tr>
<tr>
<td>9:00 – 9:40 am</td>
<td>Mathematics</td>
</tr>
<tr>
<td>9:40 – 10:20 am</td>
<td>English Language Arts</td>
</tr>
<tr>
<td>10:20 – 11:00 am</td>
<td>Science</td>
</tr>
<tr>
<td>11:00 – 12:30</td>
<td>Lunch, World Languages, and Free Time</td>
</tr>
<tr>
<td>12:30 – 1:10 pm</td>
<td>Social Studies</td>
</tr>
<tr>
<td>1:10 – 1:40</td>
<td>Afternoon Exercise</td>
</tr>
<tr>
<td>1:40 – 2:10</td>
<td>Current Events – watch the news or read the newspaper OR Language Acquisition</td>
</tr>
<tr>
<td>2:10-2:30</td>
<td>Social-Emotional Learning/Reflection/Organize for the Next Day</td>
</tr>
</tbody>
</table>

**Family Suggestions**

**Parent Suggestions**

- How can I support my student as a learner outside of school?
  - Familiarize yourself with your child’s learning calendar.
  - Encourage your child to do their best when completing tasks and assignments.
  - Contact your child’s teacher or the district’s homework hotline when you or your child have questions or need feedback.
  - Support your child in starting the daily work early in the day. Waiting until the late afternoon or evening to start work adds unnecessary stress and creates missed opportunities for collaboration and feedback.
  - Remind your child to take frequent breaks to stay focused.
  - Consider designating a dedicated workspace to maximize time on task and facilitate learning.

**Student Suggestions**

- How can I continue learning outside of school?
  - Complete work on your suggested learning calendar.
  - Put in your best effort when completing tasks and assignments.
  - Contact your teacher when you need help. Teachers are available via e-mail, your school’s online learning program or on the district’s homework hotline.
  - Let your teacher know if you have access to a phone or computer.

- How can I stay organized?
  - Start your work early. Waiting until the late afternoon or evening to start work adds unnecessary stress and creates missed opportunities for collaboration and feedback.
  - Take short breaks to increase focus and stay motivated to complete tasks on time.
  - Find a quiet place to complete your work.
## Additional Student Supports

<table>
<thead>
<tr>
<th>Individual Supports</th>
<th>See “Individualizing Support for Students” for more information on how to provide additional support to your child while at home.</th>
</tr>
</thead>
</table>
| **English Language Learners** | **Enrichment Packet**  
- Daily language learning is important! The following links/resources are available for students to access daily language learning.  
- ¡El aprendizaje diario de idiomas es importante! Los siguientes enlaces/recursos están disponibles para que los estudiantes accedan al aprendizaje diario de idiomas.  
- Kujifunza lugha ya kila siku ni muhimu! Viungo vifuatayo/rasilimali vinapatikana kwa wanafunzi kupata mafunzo ya lugha ya kila siku.  
- दैनिक भाषा सिक्ि महत्त्वपूर्ण है! तलका लिंकहरू / स्रोतहरू विद्यार्थीहरूको लागि दैनिक भाषा सिक्िने पहुँचको लागि उपलब्ध हुँ। |
| **AP** | College Board is offering free online courses on YouTube! Follow the link below to access their information. [https://apstudents.collegeboard.org/coronavirus-updates](https://apstudents.collegeboard.org/coronavirus-updates) |
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  1.2 Expected Value and Payoffs .......... 6
  1.3 Five Number Summary ................. 10
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  1.5 References .......................... 20
Statistics is hugely important for understanding, describing and predicting the world around you. Descriptive statistics is using summaries to present information that you have found to a reader. Summaries can be graphs or small groups of numbers that are easier to understand than long lists of numbers. Inferential statistics is using data to make predictions. Both inferential statistics and descriptive statistics help you understand the world around you and communicate it effectively.
1.1 Mean, Median and Mode

Learning Objectives

Here you will calculate three measures of the center of univariate data and decide which measure is best based on context.

The three measures of central tendency are mean, median, and mode. When would it make sense to use one of these measures and not the others?

Mean, Median, and Mode

With descriptive statistics, your goal is to describe the data that you find in a sample or is given in a problem. Because it would not make sense to present your findings as long lists of numbers, you summarize important aspects of the data. One important aspect of the data is the measure of central tendency, which is a measure of the “middle” value of a set of data. There are three ways to measure central tendency:

1. Use the mean, which is the arithmetic average of the data.
2. Use the median, which is the number exactly in the middle of the data. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the arithmetic average of the two most central numbers.
3. Use the mode, which is the most often occurring number in the data. If there are two or more numbers that occur equally frequently, then the data is said to be bimodal or multimodal.

Calculating the mean, median and mode is straightforward.

Take the following numbers:
3, 5, 1, 6, 8, 4, 5, 2, 7, 8, 4, 2, 1, 3, 4, 6, 7, 9, 4, 3, 2

To calculate the mean, first find the sum. The sum of all these numbers is 94 and there are 21 numbers total so the mean is \( \frac{94}{21} \approx 4.4762 \).

Note that it is common practice to round to 4 decimals in AP Statistics.

To calculate the median, first order the numbers from least to greatest. When you order the numbers from least to greatest you get:
1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9

The 11\(^{th}\) number has ten numbers to the right and ten numbers to the left so it is the median. The median is the number 4.

To find the mode, find the most frequently occurring number. the most frequently occurring number is the number 4.

What is challenging is determining when to use each measure and knowing how to interpret the data using the relationships between the three measures. Take the following situation:

Ross is with his friends and they want to play basketball. They decide to choose teams based on the number of cousins everyone has. One team will be the team with fewer cousins and the other team will be the team with more cousins.

Should they use the mean, median or mode to compute the cutoff number that will separate the two teams?
Ross and his friends should use the median number of cousins as the cutoff number because this will allow each team to have the same number of players. If there are an odd number of people playing, then the extra person will just join either team or switch in later.

**Examples**

**Example 1**

Earlier, you were asked how to determine which measure of central tendency to use. In order to decide which measure of central tendency to use, it is a good idea to calculate and interpret all three of the numbers.

For example, if someone asked you how many people can sit in the typical car, it would make more sense to use mode than to use mean. With mode, you could find out that a five person car is the most frequent car driven and determine that the answer to the question is 5. If you calculate the mean for the number of seats in all cars, you will end up with a decimal like 5.4, which makes less sense in this context.

On the other hand, if you were finding the central heights of NBA players, using mean might make a lot more sense than mode.

**Example 2**

Compute the mean, median, and mode for the following numbers.

1, 4, 5, 7, 6, 8, 0, 3, 2, 2, 3, 4, 6, 5, 7, 8, 9, 0, 6, 5, 3, 1, 2, 4, 5, 6, 7, 8, 8, 8, 4, 3, 2

The mean is 4.6061. The median is 5. The mode is 8.

**Example 3**

The cost of fresh blueberries at different times of the year are:

$2.50, $2.99, $3.20, $3.99, $4.99

If you bought blueberries regularly what would you typically pay?

The word “typically” is used instead of average to allow you to make your own choice as to whether mean, median, or mode would make the most sense. In this case, mean does make the most sense. The average cost is $3.53.

**Example 4**

Five people were called on a phone survey to respond to some political opinion questions. Two people were from the zip code 94061, one person was from the zip code 94305 and two people were from 94062.

Which measure of central tendency makes the most sense to use if you want to state where the average person was from?

None of the measures of central tendency make sense to apply to this situation. Zip codes are categorical data rather than quantitative data even though they happen to be numbers. Other examples of categorical data are hair color or gender. You could argue that mode is applicable in a broad sense, but in general remember that mean, median, and mode can only be applied to quantitative data.
Example 5

You write a computer code to produce a random number between 0 and 10 with equal probability. Unfortunately, you suspect your code doesn’t work perfectly because in your first few attempts at running the code, it produces the following numbers:

1, 9, 1, 1, 9, 2, 9, 1, 9, 9, 9, 2, 2

How would you argue using mean, median, or mode that this code is probably not producing a random number between 0 and 10 with equal probability?

This question is very similar to questions you will see when you study statistical inference.

First you would note that the mean of the data is 4.9231. If the data was truly random then the mean would probably be right around the number 5 which it is. This is not strong evidence to suggest that the random number generating code is broken.

Next you would note that the median of the data is 2. This should make you suspect that something is wrong. You would expect that the median is of random numbers between 0 and 10 to be somewhere around 5.

Lastly, you would note that the mode of the data is 9. By itself this is not strong data to suggest anything. Every sample will have to have at least one mode. What should make you suspicious, however, is the fact that only two other numbers were produced and were almost as frequent as the number 9. You would expect a greater variety of numbers to be produced.

Review

You surveyed the students in your English class to find out how many siblings each student had. Here are your results:

0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 10, 12

1. Find the mean, median, and mode of this data.

2. Why does it make sense that the mean number of siblings is greater than the median number of siblings?

3. Which measure of central tendency do you think is best for describing the typical number of siblings?

4. So far in math you have taken 10 quizzes this semester. The mean of the scores is 88.5. What is the sum of the scores?

5. Find x if 5, 9, 11, 12, 13, 14, 16, and x have a mean of 12.

6. Meera drove an average of 22 miles a day last week. How many miles did she drive last week?

7. Find x if 2, 6, 9, 8, 4, 5, 8, 1, 4, and x have a median of 5.

Calculate the mean, median, and mode for each set of numbers:

8. 11, 15, 19, 12, 21, 34, 15, 28, 24, 15, 27, 19, 20, 13, 15

9. 3, 5, 7, 5, 5, 17, 8, 9, 11, 5, 3, 7

10. -3, 0, 5, 8, 12, 4, 2, 1, 6

Calculate the mean and median for each set of numbers:

11. 12, 88, 89, 90

12. 16, 17, 19, 20, 20, 98

13. For which of the previous two questions was the median less than the mean? What in the set of numbers caused this?
14. For which of the previous two questions was the median greater than the mean? What in the set of numbers caused this?

15. In each of the sets of numbers for problems 11 and 12, there is one number that could be considered an outlier. Which numbers do you think are the outliers and why? What would happen to the mean and median if you removed the outliers?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 15.1.
1.2. Expected Value and Payoffs

Learning Objectives

Here you will apply what you know about mean and averages to weighted averages and expected value.

When playing a game of chance there are three basic elements. There is the cost to play the game (usually), the probability of winning the game, and the amount you receive if you win. If games of chance with these three elements are played repeatedly, you can use probability and averages to calculate how much you can expect to win or lose in the long run.

Consider a dice game that pays you triple your bet if you roll a six and double your bet if you roll a five. If you roll anything else you lose your bet. What is your expected return on a one dollar wager?

Expected Value and Payoffs

There are two ways to be given data, raw form and summary form. The following data represents which numbers are rolled with a standard six-sided dice:

**Data in Raw Form:**
1, 3, 5, 3, 2, 1, 2, 5, 6, 4, 5, 2, 6, 1, 4, 3, 6, 1, 2, 4, 6, 1, 3, 1, 3, 5, 6

**Data in Summary Form:**

<table>
<thead>
<tr>
<th>Number</th>
<th>Occurrence Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td><strong>Total Occurrences:</strong></td>
<td><strong>27</strong></td>
</tr>
</tbody>
</table>

Notice that the summary data indicates, for example, how many times a 1 was rolled (6 times). To calculate the total number of occurrences of data:

- In raw form: count how many data points you have
- In summary form: find the sum the occurrence column

To calculate the average:

- In raw form: find the sum of the data points and divide by the total number of occurrences.
- In summary form: find the sum of the data points by finding the sum of the product of each number and its occurrence:

\[
1 \cdot 6 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5 = 91
\]
Then, divide that sum by the total number of occurrences. In a sense, you are assigning a weight to each of the six numbers based on their frequency in your 27 trials.

The same logic of finding the average of data given in summary form applies when doing theoretical expected value for a game or a weighted average. The **expected value** is the return or cost you can expect on average, given many trials. A **weighted average** is an average that multiplies each component by a factor representing its frequency or probability. A weighted average is like a regular average except the data is often given to you in summary form.

Consider a game of chance with 4 prizes ($1, $2, $3, and $4) where each outcome has a specific probability of happening, shown in the table below:

<table>
<thead>
<tr>
<th>Number</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1</td>
<td>50%</td>
</tr>
<tr>
<td>$2</td>
<td>20%</td>
</tr>
<tr>
<td>$3</td>
<td>20%</td>
</tr>
<tr>
<td>$4</td>
<td>10%</td>
</tr>
</tbody>
</table>

Note that the probabilities must add up to 100%. In order to calculate the expected value of this game, weight the outcomes by their assigned probabilities.

\[
$1 \cdot 0.50 + $2 \cdot 0.20 + $3 \cdot 0.30 + $4 \cdot 0.10 = $2.20
\]

This means that if you were to play this game many times, your average amount of winnings should be $2.20. Note that there will be no game that you actually get $2.20, because that was none of the options. Expected value is a measure of what you should expect to get per game in the long run.

The **payoff** of a game is the expected value of the game minus the cost. If you expect to win about $2.20 on average if you play a game repeatedly and it costs only $2 to play, then the expected payoff is $0.20 per game.

In general, to find the expected value for a game or other scenario, find the sum of all possible outcomes, each multiplied by the probability of its occurrence.

**Examples**

**Example 1**

Earlier, you were asked to consider a dice game that pays you triple your bet if you roll a six and double your bet if you roll a five. For this game, the expected return on a one dollar wager is:

\[
0 \cdot \frac{2}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} = \frac{5}{6}
\]

If you spend $1 to play the game and you play the game multiple times, you can expect a return of $\frac{5}{6}$ of one dollar or about 83 cents on average.

**Example 2**

What is the expected value of an experiment with the following outcomes and corresponding probabilities?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>0.1</td>
</tr>
<tr>
<td>35</td>
<td>0.1</td>
</tr>
<tr>
<td>37</td>
<td>0.1</td>
</tr>
<tr>
<td>39</td>
<td>0.2</td>
</tr>
<tr>
<td>43</td>
<td>0.2</td>
</tr>
<tr>
<td>47</td>
<td>0.2</td>
</tr>
<tr>
<td>49</td>
<td>0.1</td>
</tr>
</tbody>
</table>
31 \cdot 0.1 + 35 \cdot 0.1 + 37 \cdot 0.1 + 39 \cdot 0.2 + 43 \cdot 0.2 + 47 \cdot 0.2 + 49 \cdot 0.1 = 41

**Example 3**

A teacher has five categories of grades that each make up a specific percentage of the final grade. Calculate Owen’s grade.

<table>
<thead>
<tr>
<th>Category</th>
<th>Weight</th>
<th>Owen’s grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quizzes and Tests</td>
<td>30%</td>
<td>78%</td>
</tr>
<tr>
<td>Homework</td>
<td>25%</td>
<td>100%</td>
</tr>
<tr>
<td>Final</td>
<td>20%</td>
<td>74%</td>
</tr>
<tr>
<td>Projects</td>
<td>20%</td>
<td>90%</td>
</tr>
<tr>
<td>Participation</td>
<td>5%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Using the concept of weighted average, weight each of Owen’s grades by the weight of the category.

\[0.78 \cdot 0.3 + 1 \cdot 0.25 + 0.74 \cdot 0.20 + 0.90 \cdot 0.20 + 1 \cdot 0.05 = 0.862\]

Owen gets an 86.2%.

**Example 4**

Courtney plays a game where she flips a coin. If the coin comes up heads she wins $2. If the coin comes up tails she loses $3. What is Courtney’s expected payoff each game?

The probability of getting heads is 50% and the probability of getting tails is 50%. Using the concept of weighted averages, you should weight winning 2 dollars and losing 3 dollars by 50% each. In this case there is no initial cost to the game.

\[2 \cdot 0.50 - 3 \cdot 0.50 = -0.50\]

This means that while sometimes she might win and sometimes she might lose, on average she is expected to lose about 50 cents per game.

**Example 5**

Paul is deciding whether or not to pay the parking meter when he is going to the movies. He knows that a parking ticket costs $30 and he estimates that there is a 40% chance that the traffic police spot his car and write him a ticket. If he chooses to pay the meter it will cost 4 dollars and he will have a 0% chance of getting a ticket.

Is it cheaper to pay the meter or risk the fine?

Since there are two possible scenarios, calculate the expected cost in each case.

- **Paying the meter**: $4 \cdot 100\% = $4
- **Risking the fine**: $0 \cdot 60\% + $30 \cdot 40\% = $12

Risking the fine has an expected cost three times that of paying the meter.

**Review**

1. Explain how to calculate expected value.
2. True or false: If the expected value of a game is $0.50, then you can expect to win $0.50 each time you play.

3. True or false: The greater the number of games played, the closer the average winnings will be to the theoretical expected value.

4. A player rolls a standard pair of dice. If the sum of the numbers is a 6, the player wins $6. If the sum of the numbers is anything else, the player has to pay $1. What is the expected value for this game?

5. What is the payoff of a slot machine that costs 25 cents to play and pays out $1 with probability 10%, $50 with probability of 1%, and $100 with probability 0.01%?

6. A slot machine pays out $1 with probability 5%, $100 with probability of 0.5%, and $1000 with probability 0.01%? If the casino wants to guarantee that they won’t lose money on this machine, how much should they charge people to play?

7. What is the expected value of an experiment with the following outcomes and corresponding probabilities?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>12</th>
<th>14</th>
<th>18</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.05</td>
<td>0.1</td>
<td>0.6</td>
<td>0.1</td>
<td>0.1</td>
<td>0.03</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Calculate the final grades for each of the students given the information in the table.

<table>
<thead>
<tr>
<th>Category</th>
<th>Weight</th>
<th>Sarah</th>
<th>Jason</th>
<th>Kimy</th>
<th>Maria</th>
<th>Kayla</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quizzes and Tests</td>
<td>30%</td>
<td>74%</td>
<td>85%</td>
<td>90%</td>
<td>80%</td>
<td>75%</td>
</tr>
<tr>
<td>Homework</td>
<td>25%</td>
<td>95%</td>
<td>40%</td>
<td>100%</td>
<td>90%</td>
<td>95%</td>
</tr>
<tr>
<td>Final</td>
<td>20%</td>
<td>68%</td>
<td>80%</td>
<td>85%</td>
<td>70%</td>
<td>50%</td>
</tr>
<tr>
<td>Projects</td>
<td>20%</td>
<td>85%</td>
<td>70%</td>
<td>95%</td>
<td>75%</td>
<td>85%</td>
</tr>
<tr>
<td>Participation</td>
<td>5%</td>
<td>95%</td>
<td>100%</td>
<td>100%</td>
<td>80%</td>
<td>60%</td>
</tr>
</tbody>
</table>

8. What is Sarah’s final grade?

9. What is Jason’s final grade?

10. What is Kimy’s final grade?

11. What is Maria’s final grade?

12. What is Kayla’s final grade?

13. Look back at the grades and final grades for the five students. Do the grades seem fair to you given how each student performed in each of the areas? Do you think the category weights should be changed?

14. You are in charge of a booth for a game at the fair. In the game, players pick a card at random from the deck. If the card is a J, Q, or K, the player wins $5. What is the minimum amount you should charge in order to feel confident you will make a profit by the end of the fair?

15. Make up your own game that has at least 2 possible outcomes with an expected payoff of $0.50.

16. Explain why it makes sense for a casino to consider the concept of expected value when designing their games.

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 15.2.
1.3 Five Number Summary

Learning Objectives

Here you will calculate quartiles and produce five number summaries for data sets.

When given a long list of numbers, it is useful to summarize the data. One way to summarize the data is to give the lowest number, the highest number and the middle number. In addition to these three numbers it is also useful to give the median of the lower half of the data and the median of the upper half of the data. These five numbers give a very concise summary of the data.

What is the five number summary of the following data?
0, 0, 1, 2, 63, 61, 27, 13

The Five Number Summary

Suppose you have ordered data with \( m \) observations. The rank of each observation is shown by its index. The rank of an observation is the number of observations that are less than or equal to the value of that observation.

\[ y_1 \leq y_2 \leq y_3 \leq \cdots \leq y_m \]

In data sets that are large enough, you can divide the numbers into four parts called quartiles. The quartiles of interest are the first quartile, \( Q_1 \), the second quartile, \( Q_2 \), and the third quartile \( Q_3 \). The second quartile, \( Q_2 \), is defined to be the median of the data. The first quartile, \( Q_1 \), is defined to be the median of the lower half of the data. The third quartile, \( Q_3 \), is similarly defined to be the median of the upper half of the data.

These three numbers in addition to the minimum and maximum values are the five number summary. Note that there are variations of the five number summary that you can study in a statistics course.

Take the following data:
2, 7, 17, 19, 25, 26, 26, 32

There are 8 observations total in this set of data.

- Lowest value (minimum) : 2
- \( Q_1 : \frac{7+17}{2} = 12 \) (Note that this is the median of the first half of the data - 2, 7, 17, 19)
- \( Q_2 : \frac{19+25}{2} = 22 \) (Note that this is the median of the full set of data)
- \( Q_3 : 26 \) (Note that this is the median of the second half of the data - 25, 26, 26, 32)
- Upper value (maximum) : 32

The 5 number summary is 2, 12, 22, 26, 32.
Examples

Example 1

Earlier you were asked to compute the five number summary for 0, 0, 1, 2, 63, 61, 27, 13. It helps to order the data.

0, 0, 1, 2, 13, 27, 61, 63

- Since there are 8 observations, the median is the average of the 4th and 5th observations: \( \frac{2+13}{2} = 7.5 \)
- The lowest observation is 0.
- The highest observation is 63.
- The middle of the lower half is \( \frac{0+1}{2} = 0.5 \)
- The middle of the upper half is \( \frac{27+61}{2} = 44 \)

The five number summary is 0, 0.5, 7.5, 44, 63

Example 2

Create a set of data that meets the following five number summary:

\{2, 5, 9, 18, 20\}

Suppose there are 8 data points. The lowest point must be 2 and the highest point must be 20. The middle two points must average to be 9 so they could be 8 and 10. The second and third points must average to be 5 so they could be 4 and 6. The sixth and seventh points need to average to be 18 so they could be 18 and 18. Here is one possible answer:

2, 4, 6, 8, 10, 18, 18, 20

Example 3

Compute the five number summary for the following data:

1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15

There are 20 observations.

- Lower : 1
- \( Q_1 : \frac{2+3}{2} = 2.5 \)
- \( Q_2 : \frac{4.5+5.5}{2} = 5 \)
- \( Q_3 : \frac{6+7}{2} = 6.5 \)
- Upper : 15

Example 4

Compute the five number summary for the following data:

4, 8, 11, 11, 12, 14, 16, 20, 21, 25

There are 10 observations total in this set of data.

- Lowest value (minimum) : 4
1.3. Five Number Summary

- \(Q_1\) : 11 (Note that this is the median of the first half of the data - 4, 8, 11, 11, 12)
- \(Q_2\) : \(\frac{12+14}{2} = 13\) (Note that this is the median of the full set of data)
- \(Q_3\) : 20 (Note that this is the median of the second half of the data - 14, 16, 20, 21, 25)
- Upper value (maximum) : 25

The five number summary is 4, 11, 13, 20, 25.

**Example 5**

Compute the five number summary for the following data:

3, 7, 10, 14, 19, 19, 23, 27, 29

There are 9 observations total. To calculate \(Q_1\) and \(Q_3\), you should include the median in both the lower half and upper half calculations.

- Lowest value (minimum) : 3
- \(Q_1\) : 10 (this is the median of 3, 7, 10, 14, 19)
- \(Q_2\) : 19
- \(Q_3\) : 23 (this is the median of 19, 19, 23, 27, 29)
- Upper value (maximum) : 29

The five number summary is 3, 10, 19, 23, 29.

**Review**

Compute the five number summary for each of the following sets of data:

1. 0.16, 0.08, 0.27, 0.20, 0.22, 0.32, 0.25, 0.18, 0.28, 0.27
2. 77, 79, 80, 86, 87, 87, 94, 99
3. 79, 53, 82, 91, 87, 98, 80, 93
4. 91, 85, 76, 86, 96, 51, 68, 92, 85, 72, 66, 88, 93, 82, 84
5. 335, 233, 185, 392, 235, 518, 281, 208, 318
6. 38, 33, 41, 37, 54, 39, 38, 71, 49, 48, 42, 38
7. 3, 7, 8, 5, 12, 14, 21, 13, 18
8. 6, 22, 11, 25, 16, 26, 28, 37, 37, 38, 33, 40, 34, 39, 23, 11, 48, 49, 8, 26, 18, 17, 27, 14
9. 9, 10, 12, 13, 10, 14, 8, 10, 12, 6, 8, 11, 12, 9, 11, 10, 15, 10, 8, 8, 12, 10, 14, 10, 9, 7, 5, 11, 15, 8, 9, 17, 12, 12, 13, 7, 14, 6, 17, 11, 15, 10, 13, 9, 7, 12, 13, 10, 12
10. 49, 57, 53, 54, 49, 67, 51, 57, 56, 59, 57, 50, 49, 52, 53, 50, 58
11. 18, 20, 24, 21, 5, 23, 19, 22
13. 13, 15, 19, 14, 26, 17, 12, 42, 18
14. 25, 33, 55, 32, 17, 19, 15, 18, 21
15. 149, 123, 126, 122, 129, 120

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 15.3.
1.4 Graphic Displays of Data

Learning Objectives

Here you will display data using bar charts, histograms, pie charts and boxplots.

Two common types of graphic displays are bar charts and histograms. Both bar charts and histograms use vertical or horizontal bars to represent the number of data points in each category or interval. The main difference graphically is that in a bar chart there are spaces between the bars and in a histogram there are not spaces between the bars. Why does this subtle difference exist and what does it imply about graphic displays in general?

Displaying Data

It is often easier for people to interpret relative sizes of data when that data is displayed graphically. Note that a categorical variable is a variable that can take on one of a limited number of values and a quantitative variable is a variable that takes on numerical values that represent a measurable quantity. Examples of categorical variables are tv stations, the state someone lives in, and eye color while examples of quantitative variables are the height of students or the population of a city. There are a few common ways of displaying data graphically that you should be familiar with.

Pie Chart

A pie chart shows the relative proportions of data in different categories. Pie charts are excellent ways of displaying categorical data with easily separable groups. The following pie chart shows six categories labeled A – F. The size of each pie slice is determined by the central angle. Since there are 360° in a circle, the size of the central angle \( \theta_A \) of category A can be found by:

\[
\theta_A = \frac{\text{# data points in category } A}{\text{Total number of data points}} \times 360
\]
1.4. Graphic Displays of Data

Bar Chart

A **bar chart** displays frequencies of categories of data. The bar chart below has 5 categories, and shows the TV channel preferences for 53 adults. The horizontal axis could have also been labeled News, Sports, Local News, Comedy, Action Movies. The reason why the bars are separated by spaces is to emphasize the fact that they are categories and not continuous numbers. For example, just because you split your time between channel 8 and channel 44 does not mean on average you watch channel 26. Categories can be numbers so you need to be very careful.

![Bar Chart Image]

Histogram

A **histogram** displays frequencies of quantitative data that has been sorted into intervals. The following is a histogram that shows the heights of a class of 53 students. Notice the largest category is 56-60 inches with 18 people.

![Histogram Image]
Boxplot

A boxplot (also known as a box and whiskers plot) is another way to display quantitative data. It displays the five-number summary (minimum, $Q_1$, median, $Q_3$, maximum). The box can either be vertically or horizontally displayed depending on the labeling of the axis. The box does not need to be perfectly symmetrical because it represents data that might not be perfectly symmetrical.

Examples

Example 1

Earlier, you were asked about the difference between histograms and bar charts. The reason for the space in bar charts but no space in histograms is bar charts graph categorical variables while histograms graph quantitative variables. It would be extremely improper to forget the space with bar charts because you would run the risk of implying a spectrum from one side of the chart to the other. Note that in the bar chart where TV stations were shown, the station numbers were not listed horizontally in order by size. This was to emphasize the fact that the stations were categories.

Example 2

Create a boxplot of the following numbers in your calculator.

8.5, 10.9, 9.1, 7.5, 7.2, 6, 2.3, 5.5

Enter the data into $L_1$ by going into the Stat menu.

Then turn the statplot on and choose boxplot.
1.4. Graphic Displays of Data

Use Zoomstat to automatically center the window on the boxplot.

Example 3

Create a pie chart to represent the preferences of 43 hungry students.

- Other - 5
- Burritos - 7
- Burgers - 9
- Pizza - 22
Example 4

Create a bar chart representing the preference for sports of a group of 23 people.

- Football - 12
- Baseball - 10
- Basketball - 8
- Hockey - 3

![Bar chart showing sports preferences](chart1.png)

Example 5

Create a histogram for the income distribution of 200 million people.

- Below $50,000 is 100 million people
- Between $50,000 and $100,000 is 50 million people
- Between $100,000 and $150,000 is 40 million people
- Above $150,000 is 10 million people

![Histogram showing income distribution](chart2.png)
1.4. Graphic Displays of Data

Review

1. What types of graphs show categorical data?
2. What types of graphs show quantitative data?

A math class of 30 students had the following grades:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Number of Students with Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1.7:

3. Create a bar chart for this data.
4. Create a pie chart for this data.
5. Which graph do you think makes a better visual representation of the data?

A set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98

6. Create a histogram for this data. Use your best judgment to decide what the intervals should be.
7. Find the five number summary for this data.
8. Use the five number summary to create a boxplot for this data.
9. Describe the data shown in the boxplot below.

10. Describe the data shown in the histogram below.
A math class of 30 students has the following eye colors:

**Table 1.8:**

<table>
<thead>
<tr>
<th>Grade</th>
<th>Number of Students with Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brown</td>
<td>20</td>
</tr>
<tr>
<td>Blue</td>
<td>5</td>
</tr>
<tr>
<td>Green</td>
<td>3</td>
</tr>
<tr>
<td>Other</td>
<td>2</td>
</tr>
</tbody>
</table>

11. Create a bar chart for this data.

12. Create a pie chart for this data.

13. Which graph do you think makes a better visual representation of the data?

14. Suppose you have data that shows the breakdown of registered republicans by state. What types of graphs could you use to display this data?

15. From which types of graphs could you obtain information about the spread of the data? Note that spread is a measure of how spread out all of the data is.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 15.4.

You started by working with univariate data and learned how to display it graphically and summarize it numerically. You learned how to calculate mean, median, mode and variance, and when to use each. You also explored bivariate data and used the regression capabilities of your calculator to create mathematical models for real world phenomenon.
5. CK-12 Foundation. . CCSA
6. CK-12 Foundation. . CCSA
8. CK-12 Foundation. . CCSA
9. CK-12 Foundation. . CCSA
10. CK-12 Foundation. . CCSA
11. CK-12 Foundation. . CCSA
12. CK-12 Foundation. . CCSA
## Contents

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1.2 Compound Interest per Year ........................................... 6  
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1.6 Annuities ............................................................... 20  
1.7 Annuities for Loans .................................................... 25
Here you will review concepts of exponential growth and geometric series with a focus on the relationship between time and money.
Learning Objectives

Here you’ll learn to calculate the effect of time on the balance of a savings account growing by simple interest.

The basic concept of interest is that a dollar today will be worth more than a dollar next year. If a person deposits $100 into a bank account today at 6% simple interest, then in one year the bank owes the person that $100 plus a few dollars more. If the person decides to leave it in the account and keep earning the interest, then after two years the bank would owe the person even more money. How much interest will the person earn each year? How much money will the person have after two years?

Simple Interest

Simple interest is defined as interest that only accumulates on the initial money deposited in the account. This initial money is called the principal. In the real world, most companies do not use simple interest because it is considered too simple and instead use compound interest which compounds on itself. You will practice with simple interest here because it introduces the concept of the time value of money and that a dollar saved today is worth slightly more than a dollar in one year.

The formula for simple interest has 4 variables and all the problems and examples will give 3 and your job will be to find the unknown quantity using rules of Algebra.

\[ FV = PV(1 + t \cdot i) \]

Let’s say Linda invested $1,000 for her child’s college education and she saved it for 18 years at a bank which offered 5% simple interest. To find out how much she has at the end of 18 years, first identify known and unknown quantities.

\[
\begin{align*}
PV &= $1,000 \\
 t &= 18 \text{ years} \\
i &= 0.05 \\
FV &= \text{unknown so you will use } x
\end{align*}
\]

Then substitute the values into the formula and solve to find the future value.
Linda initially had $1,000, but 18 years later with the effect of 5% simple interest, that money grew to $1,900.

### Examples

#### Example 1

Earlier, you were asked about the how much a person who deposits $100 today at 6% simple interest will have in one year and in two years. That person will have have $106 in one year and $112 in two years.

#### Example 2

Tory put $200 into a bank account that earns 8% simple interest. How much interest does Tory earn each year and how much does she have at the end of 4 years?

First you will focus on the first year and identify known and unknown quantities.

\[ PV = 200 \]
\[ t = 1 \text{ year} \]
\[ i = 0.08 \]
\[ FV = \text{unknown so we will use } x \]

Second, you will substitute the values into the formula and solve to find the future value.

The third thing you need to do is interpret and organize the information. Tory had $200 to start with and then at the end of one year she had $216. The additional $16 is interest she has earned that year. Since the account is simple interest, she will keep earning $16 dollars every year because her principal remains at $200. The $16 of interest earned that first year just sits there earning no interest of its own for the following three years.
At the end of 4 years, Tory will have $264 on her account. $64 will be interest. She earned $16 in interest each year.

**Example 3**

Amy has $5000 to save and she wants to buy a car for $10,000. For how many years will she need to save if she earns 10% simple interest? On the other hand, what will the simple interest rate need to be if she wants to save enough money in 15 years?

Notice that there are two separate problems. Let’s start with the first problem and identify known and unknown quantities.

\[ PV = 5000, \quad FV = 10000, \quad i = 0.10, \quad t =? \]

Now substitute and solve for \( t \).

Now let’s focus on the second problem and go through the process of identifying known and unknown quantities, substituting and solving.

To answer the first question, Amy would need to save for 10 years getting a simple interest rate of 10%. For the second question, she would need to save for 15 years at a simple interest rate of about 6.667%.

**Example 4**

How long will it take $3,000 to grow to $4,000 at 4% simple interest?

\[ PV = 3000, \quad t =?, \quad i = 0.04, \quad FV = 4000 \]
Example 5

What starting balance grows to $5,000 in 5 years with 10% simple interest?

\[ PV = ?, \quad FV = 5,000, \quad t = 5, \quad i = 0.10 \]

Review

1. How much will a person have at the end of 8 years if they invest $3,000 at 4.5% simple interest?
2. How much will a person have at the end of 6 years if they invest $2,000 at 3.75% simple interest?
3. How much will a person have at the end of 12 years if they invest $1,500 at 7% simple interest?
4. How much interest will a person earn if they invest $10,000 for 10 years at 5% simple interest?
5. How much interest will a person earn if they invest $2,300 for 49 years at 3% simple interest?
6. How long will it take $2,000 to grow to $5,000 at 3% simple interest?
7. What starting balance grows to $12,000 in 8 years with 10% simple interest?
8. Suppose you have $3,000 and want to have $35,000 in 25 years. What simple interest rate will you need?
9. How long will it take $1,000 to grow to $4,000 at 8% simple interest?
10. What starting balance grows to $9,500 in 4 years with 6.5% simple interest?
11. Suppose you have $1,500 and want to have $8,000 in 15 years. What simple interest rate will you need?
12. Suppose you have $800 and want to have $6,000 in 45 years. What simple interest rate will you need?
13. What starting balance grows to $2,500 in 2 years with 1.5% simple interest?
14. Suppose you invest $4,000 which earns 5% simple interest for the first 12 years and then 8% simple interest for the next 8 years. How much money will you have after 20 years?
15. Suppose you invest $10,000 which earns 2% simple interest for the first 8 years and then 5% simple interest for the next 7 years. How much money will you have after 15 years?

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.1.

Principal is the amount initially deposited into the account. Notice the spelling is principal, not principle.

Interest is the conversion of time into money.
1.2 Compound Interest per Year

Learning Objectives

Here you’ll explore how to compute an investment’s growth given time and a compound interest rate.

If a person invests $100 in a bank with 6% simple interest, they earn $6 in the first year and $6 again in the second year totaling $112. If this was really how interest operated with most banks, then someone clever may think to withdraw the $106 after the first year and immediately reinvest it. That way they earn 6% on $106. At the end of the second year, the clever person would have earned $6 like normal, plus an extra .36 cents totaling $112.36. Thirty six cents may seem like not very much, but how much more would a person earn if they saved their $100 for 50 years at 6% compound interest rather than at just 6% simple interest?

**Compound Interest Per Year**

**Compound interest** allows interest to grow on interest. As with simple interest, $PV$ is defined as present value, $FV$ is defined as future value, $i$ is the interest rate, and $t$ is time. The formulas for simple and compound interest look similar, so be careful when reading problems in determining whether the interest rate is simple or compound. The following table shows the amount of money in an account earning compound interest over time:

**Table 1.2:**

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount Ending in Account</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$FV = PV(1 + i)$</td>
</tr>
<tr>
<td>2</td>
<td>$FV = PV(1 + i)^2$</td>
</tr>
<tr>
<td>3</td>
<td>$FV = PV(1 + i)^3$</td>
</tr>
<tr>
<td>4</td>
<td>$FV = PV(1 + i)^4$</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>$FV = PV(1 + i)^t$</td>
</tr>
</tbody>
</table>

For now you should assume that you are compounding the interest once a year or annually. An account with a present value of $PV$ that earns compound interest at $i$ percent annually for $t$ years has a future value of $FV$ shown below:

$$FV = PV(1 + i)^t$$

Applying this formula for years 1, 2, 3, and 4 for an initial deposit of $100 at 3% compound interest, you would get the following results:

$PV = 100$, $i = 0.03$, $t = 1, 2, 3$ and 4, $FV =$?

**Table 1.3:**

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount ending in Account</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$FV = 100(1 + 0.03) = 103.00$</td>
</tr>
</tbody>
</table>
### Table 1.3: (continued)

<table>
<thead>
<tr>
<th></th>
<th>( FV = 100(1 + 0.03)^t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( = 106.09 )</td>
</tr>
<tr>
<td>3</td>
<td>( = 109.27 \approx 109 )</td>
</tr>
<tr>
<td>4</td>
<td>( = 112.55 \approx 112 )</td>
</tr>
</tbody>
</table>

Calculator shortcut: When doing repeated calculations that are just 1.03 times the result of the previous calculation, use the <ANS> button to create an entry that looks like <Ans*1.03>. Then, pressing enter repeatedly will rerun the previous entry producing the values on the right.

### Examples

#### Example 1

Earlier you were introduced to a concept problem contrasting $100 for 50 years at 6% compound interest versus 6% simple. Now you can calculate how much more powerful compound interest is.

\( PV = 100, \ t = 50, \ i = 6\%, \ FV = ? \)

Simple interest:
\[
FV = PV(1 + t \cdot i) = 100(1 + 50 \cdot 0.06) = 400
\]

Compound interest:
\[
FV = PV(1 + i)^t = 100(1 + 0.06)^{50} \approx 1,842.02
\]

It is remarkable that simple interest grows the balance of the account to $400 while compound interest grows it to about $1,842.02. The additional money comes from interest growing on interest repeatedly.

#### Example 2

How much will Kyle have in a savings account if he saves $3,000 at 4% compound interest for 10 years?

\( PV = 3,000, \ i = 0.04, \ t = 10 \text{ years}, \ FV = ? \)

\[
FV = PV(1 + i)^t = 3000(1 + 0.04)^{10} \approx 4,440.73
\]

#### Example 3

How long will it take money to double if it is in an account earning 8% compound interest?

There are two ways you can solve this problem, through estimation or through computation.

Estimation Solution: The rule of 72 is an informal means of estimating how long it takes money to double. It is useful because it is a calculation that can be done mentally that can yield surprisingly accurate results. This can be very helpful when doing complex problems to check and see if answers are reasonable. The rule of 72 simply states \( \frac{72}{i} \approx t \) where \( i \) is written as an integer (i.e. 8% would just be 8).

In this case \( \frac{72}{8} = 9 \approx t \), so it will take about 9 years.
Exact Solution: Since there is no initial value you are just looking for some amount to double. You can choose any amount for the present value and double it to get the future value even though specific numbers are not stated in the problem. Here you should choose 100 for $PV$ and 200 for $FV$.

$PV = 100, FV = 200, i = 0.08, t = ?$

\[
FV = PV(1 + i)^t \\
200 = 100(1 + 0.08)^t \\
2 = (1.08)^t \\
\ln 2 = \ln 1.08^t \\
\ln 2 = t \cdot \ln 1.08 \\
t = \frac{\ln 2}{\ln 1.08} \approx 9.00646
\]

It will take just over 9 years for money (any amount) to double at 8%. This is extraordinarily close to your estimation and demonstrates how powerful the Rule of 72 can be in estimation.

**Example 4**

How long will it take money to double at 6% compound interest? Estimate using the rule of 72 and also find the exact answer.

Estimate: \(\frac{72}{6} = 12 \approx \) years it will take to double

$PV = 100, FV = 200, i = 0.06, t = ?$

\[
200 = 100(1 + 0.06)^t \\
2 = (1.06)^t \\
\ln 2 = \ln 1.06^t = t \cdot \ln 1.06 \\
t = \frac{\ln 2}{\ln 1.06} \approx 11.89 \text{ years}
\]

**Example 5**

What compound interest rate is needed to grow $100 to $120 in three years?

$PV = 100, FV = 120, t = 3, i =$

\[
FV = PV(1 + i)^t \\
120 = 100(1 + i)^3 \\
[1.2]^\frac{1}{3} = [(1 + i)^3]^\frac{1}{3} \\
[1.2]^\frac{1}{3} = 1 + i \\
i = 1.2^\frac{1}{3} - 1 \approx 0.06266
\]
Review

For problems 1-10, find the missing value in each row using the compound interest formula.

**Table 1.4:**

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>( PV )</th>
<th>( FV )</th>
<th>( t )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$1,000</td>
<td></td>
<td>7</td>
<td>1.5%</td>
</tr>
<tr>
<td>2.</td>
<td>$1,575</td>
<td>$2,250</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>$4,500</td>
<td>$5,534.43</td>
<td>3%</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td></td>
<td>$10,000</td>
<td>12</td>
<td>2%</td>
</tr>
<tr>
<td>5.</td>
<td>$1,670</td>
<td>$3,490</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>$17,000</td>
<td>$40,000</td>
<td>25</td>
<td>5%</td>
</tr>
<tr>
<td>7.</td>
<td>$10,000</td>
<td>$17,958.56</td>
<td></td>
<td>8%</td>
</tr>
<tr>
<td>8.</td>
<td></td>
<td>$50,000</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td></td>
<td>$1,000,000</td>
<td>40</td>
<td>6%</td>
</tr>
<tr>
<td>10.</td>
<td>$10,000</td>
<td></td>
<td>50</td>
<td>7%</td>
</tr>
</tbody>
</table>

11. How long will it take money to double at 4% compound interest? Estimate using the rule of 72 and also find the exact answer.

12. How long will it take money to double at 3% compound interest? Estimate using the rule of 72 and also find the exact answer.

13. Suppose you have $5,000 to invest for 10 years. How much money would you have in 10 years if you earned 4% simple interest? How much money would you have in 10 years if you earned 4% compound interest?

14. Suppose you invest $4,000 which earns 5% compound interest for the first 12 years and then 8% compound interest for the next 8 years. How much money will you have after 20 years?

15. Suppose you invest $10,000 which earns 2% compound interest for the first 8 years and then 5% compound interest for the next 7 years. How much money will you have after 15 years?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 13.2.
1.3 Compound Interest per Period

Learning Objectives

Here you’ll learn to compute future values with interest that accumulates semi-annually, monthly, daily, etc.

Clever Carol went to her bank which was offering 12% interest on its savings account. She asked very nicely if instead of having 12% at the end of the year, if she could have 6% after the first 6 months and then another 6% at the end of the year. Carol and the bank talked it over and they realized that while the account would still seem like it was getting 12%, Carol would actually be earning a higher percentage. How much more will Carol earn this way?

Compound Interest Per Period of Time

Consider a bank that compounds and adds interest to accounts $k$ times per year. If the original percent offered is 12% then in one year that interest can be compounded:

- Once, with 12% at the end of the year ($k = 1$)
- Twice (semi-annually), with 6% after the first 6 months and 6% after the last six months ($k = 2$)
- Four times (quarterly), with 3% at the end of each 3 months ($k = 4$)
- Twelve times (monthly), with 1% at the end of each month ($k = 12$)

The intervals could even be days, hours or minutes. This is called the length of the **compounding period**. The number of compounding periods is how often interest is compounded. When intervals become small so does the amount of interest earned in that period, but since the intervals are small there are more of them. This effect means that there is a much greater opportunity for interest to compound.

**Nominal interest** is a number that resembles an interest rate, but it really is a sum of compound interest rates. A nominal rate of 12% compounded monthly is really 1% compounded 12 times. The formula for interest compounding $k$ times per year for $t$ years at a nominal interest rate $i$ with present value $PV$ and future value $FV$ is:

$$FV = PV \left(1 + \frac{i}{k}\right)^{kt}$$

As with simple interest and compound interest, the nominal rate of interest is represented with the letter $i$ in this formula, but the resulting rate is computed differently. A nominal rate of 12% may actually yield more than 12%.

Let’s apply the formula above to an investment of $300 at a rate of 12% compounded monthly. If you wanted to know the amount of money the person would have after 4 years, you would take the following steps:

$$FV = ?, \quad PV = 300, \quad t = 4, \quad k = 12, \quad i = 0.12$$

$$FV = PV \left(1 + \frac{i}{k}\right)^{kt} = 300 \left(1 + \frac{0.12}{12}\right)^{12 \cdot 4} \approx 483.67$$

Note: A very common mistake when typing the values into a calculator is using an exponent of 12 and then multiplying the whole quantity by 4 instead of using an exponent of $(12 \cdot 4) = 48$.

Examples

**Example 1**

Earlier, you were asked about Clever Carol and the difference in amount of money she would have if her interest was compounded once a year versus twice a year. If Clever Carol earned the 12% at the end of the year she would
earn $12 in interest in the first year. If she compounds it \( k = 2 \) times per year then she will end up earning:

\[
FV = PV \left(1 + \frac{i}{k}\right)^{kt} = 100 \left(1 + \frac{12}{2}\right)^{2 \cdot 1} = $112.36
\]

**Example 2**

How many years will Matt need to invest his money at 6% compounded daily \( (k = 365) \) if he wants his $3,000 to grow to $5,000?

\[
FV = 5,000, \, PV = 3,000, \, k = 365, \, i = 0.06, \, t = ?
\]

\[
FV = PV \left(1 + \frac{i}{k}\right)^{kt}
\]

\[
5,000 = 3,000 \left(1 + \frac{0.06}{365}\right)^{365t}
\]

\[
\frac{5}{3} = \left(1 + \frac{0.06}{365}\right)^{365t}
\]

\[
\ln \frac{5}{3} = \ln \left(1 + \frac{0.06}{365}\right)^{365t}
\]

\[
\ln \frac{5}{3} = 365t \cdot \ln \left(1 + \frac{0.06}{365}\right)
\]

\[
t = \frac{\ln \frac{5}{3}}{365 \cdot \ln \left(1 + \frac{0.06}{365}\right)} = 8.514 \text{ years}
\]

**Example 3**

What nominal interest rate compounded quarterly doubles money in 5 years?

\( FV = 200, \, PV = 100, \, k = 4, \, i = ?, \, t = 5 \)

**Example 4**

How much will Steve have in 8 years if he invests $500 in a bank that offers 8% compounded quarterly?

\( PV = 500, \, t = 8, \, i = 8\%, \, FV = ?, \, k = 4 \)

\[
FV = PV \left(1 + \frac{i}{k}\right)^{kt} = 500 \left(1 + \frac{0.08}{4}\right)^{4 \cdot 8} = $942.27
\]

**Example 5**

How many years will Mark need to invest his money at 3% compounded weekly \( (k = 52) \) if he wants his $100 to grow to $400?
1.3. Compound Interest per Period

\[ FV = 400, \ PV = 100, \ k = 52, \ i = 0.03, \ t = ? \]

\[
FV = PV \left(1 + \frac{i}{k}\right)^{kt}
\]

\[
400 = 100 \left(1 + \frac{0.03}{52}\right)^{52t}
\]

\[
\ln 4 = \ln \left(1 + \frac{0.03}{52}\right)^{52t}
\]

\[
t = \frac{\ln 4}{52 \cdot \ln \left(1 + \frac{0.03}{52}\right)} = 46.22 \text{ years}
\]

**Review**

1. What is the length of a compounding period if \( k = 12 \)?
2. What is the length of a compounding period if \( k = 365 \)?
3. What would the value of \( k \) be if interest was compounded every hour?
4. What would the value of \( k \) be if interest was compounded every minute?
5. What would the value of \( k \) be if interest was compounded every second?

For problems 6-15, find the missing value in each row using the compound interest formula.

**Table 1.5:**

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>PV</th>
<th>FV</th>
<th>t</th>
<th>i</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$1,000</td>
<td></td>
<td>7</td>
<td>1.5%</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>$1,575</td>
<td>$2,250</td>
<td>5</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$4,000</td>
<td>$5,375.67</td>
<td></td>
<td>3%</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>$10,000</td>
<td>12</td>
<td>2%</td>
<td>365</td>
</tr>
<tr>
<td>10</td>
<td>$10,000</td>
<td></td>
<td>50</td>
<td>7%</td>
<td>52</td>
</tr>
<tr>
<td>11</td>
<td>$1,670</td>
<td>$3,490</td>
<td>10</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>12</td>
<td>$17,000</td>
<td>$40,000</td>
<td>25</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>$12,000</td>
<td></td>
<td>3</td>
<td>5%</td>
<td>365</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>$50,000</td>
<td>30</td>
<td>8%</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>$1,000,000</td>
<td></td>
<td>40</td>
<td>6%</td>
<td>2</td>
</tr>
</tbody>
</table>

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 13.3.
1.4 Continuous Interest

Learning Objectives

Here you’ll learn to use the force of interest to compute future values when interest is being compounded continuously.

Clever Carol realized that she makes more money when she convinces the bank to give her 12% in two chunks of 6% than only one time at 12%. Carol knew she could convince them to give her 1% at the end of each month for a total of 12% which would be even more than the two chunks of 6%. As Carol makes the intervals smaller and smaller, does she earn more and more money from the bank? Does this extra amount ever stop or does it keep growing forever?

Continuous Interest

Calculus deals with adding up an infinite number of infinitely small amounts. Using calculus, we can derive the value $e$ to help us understand what happens as $k$, the number of compounding periods, approaches infinity. The number $e$ is used frequently in finance and other fields to represent this type of continuous growth.

$e \approx \left(1 + \frac{1}{k}\right)^k \approx 2.71828 \ldots$ as $k$ approaches infinity

This means that even when there are an infinite number of infinitely small compounding periods, there will be a limit on the interest earned in a year. The term for infinitely small compounding periods is continuous compounding. A continuously compounding interest rate is the rate of growth proportional to the amount of money in the account at every instantaneous moment in time. It is equivalent to infinitely many but infinitely small compounding periods.

The formula for finding the future value of a present value invested at a continuously compounding interest rate $r$ for $t$ years is:

$FV = PV \cdot e^{rt}$

Applying this formula, you can determine what the future value of $360 invested for 6 years at a continuously compounding rate of 5% is.

$FV = ?, \ PV = 360, \ r = 0.05, \ t = 6$

$FV = PV \cdot e^{rt} = 360e^{0.05 \cdot 6} = 360e^{0.30} \approx 485.95$

MEDIA

Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/57214
## Examples

### Example 1

Earlier, you were asked to compare the amount of money Clever Carol would make using different rates of compounding. Clever Carol could calculate the returns on each of the possible compounding periods for one year.

For once per year, $k = 1$:

$$FV = PV(1+i)^t = 100(1+0.12)^1 = 112$$

For twice per year, $k = 2$:

$$FV = PV(1+i)^t = 100 \left(1 + \frac{0.12}{2}\right)^2 = 112.36$$

For twelve times per year, $k = 12$:

$$FV = PV(1+i)^t = 100 \left(1 + \frac{0.12}{12}\right)^{12} \approx 112.68$$

At this point Carol might notice that while she more than doubled the number of compounding periods, she did not more than double the extra pennies. The growth slows down and approaches the continuously compounded growth result.

For continuously compounding interest:

$$FV = PV \cdot e^{rt} = 100 \cdot e^{0.12 \cdot 1} \approx 112.75$$

No matter how many times Clever Carol might convince her bank to compound the 12% over the course of each year, the most she can earn from the original $100 is around $12.75 in interest.

### Example 2

What is the continuously compounding rate that will grow $100 into $250 in just 2 years? 

$PV = 100, FV = 250, r = ?, t = 2$

$$FV = PV \cdot e^{rt}$$

$$250 = 100 \cdot e^{2r}$$

$$2.5 = e^{2r}$$

$$\ln 2.5 = 2r$$

$$r = \frac{\ln 2.5}{2} \approx 0.4581 = 45.81\%$$

### Example 3

What amount invested at 7% continuously compounding yields $1,500 after 8 years? 

$PV = ?, FV = 1,500, t = 8, r = 0.07$

$$FV = PV \cdot e^{rt}$$

$$1,500 = PV \cdot e^{0.07\cdot 8}$$

$$PV = \frac{1,500}{e^{0.07\cdot 8}} \approx 856.81$$
Example 4

What is the future value of $500 invested for 8 years at a continuously compounding rate of 9%?

\[ FV = 500e^{8 \cdot 0.09} \approx 1027.22 \]

Example 5

What is the continuously compounding rate which grows $27 into $99 in just 4 years?

\[ 99 = 27e^{4r} \]

Solving for \( r \) yields: \( r = 0.3248 = 32.48\% \)

Review

For problems 1-10, find the missing value in each row using the continuously compounding interest formula.

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>( PV )</th>
<th>( FV )</th>
<th>( t )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$1,000</td>
<td></td>
<td>7</td>
<td>1.5%</td>
</tr>
<tr>
<td>2.</td>
<td>$1,575</td>
<td>$2,250</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>$4,500</td>
<td>$5,500</td>
<td></td>
<td>3%</td>
</tr>
<tr>
<td>4.</td>
<td></td>
<td>$10,000</td>
<td>12</td>
<td>2%</td>
</tr>
<tr>
<td>5.</td>
<td>$1,670</td>
<td>$3,490</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>$17,000</td>
<td>$40,000</td>
<td></td>
<td>5%</td>
</tr>
<tr>
<td>7.</td>
<td>$10,000</td>
<td>$18,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td></td>
<td>$50,000</td>
<td>30</td>
<td>8%</td>
</tr>
<tr>
<td>9.</td>
<td></td>
<td>$1,000,000</td>
<td>40</td>
<td>6%</td>
</tr>
<tr>
<td>10.</td>
<td>$10,000</td>
<td></td>
<td>50</td>
<td>7%</td>
</tr>
</tbody>
</table>

11. How long will it take money to double at 4% continuously compounding interest?
12. How long will it take money to double at 3% continuously compounding interest?
13. Suppose you have $6,000 to invest for 12 years. How much money would you have in 10 years if you earned 3% simple interest? How much money would you have in 10 years if you earned 3% continuously compounding interest?
14. Suppose you invest $2,000 which earns 5% continuously compounding interest for the first 12 years and then 8% continuously compounding interest for the next 8 years. How much money will you have after 20 years?
15. Suppose you invest $7,000 which earns 1.5% continuously compounding interest for the first 8 years and then 6% continuously compounding interest for the next 7 years. How much money will you have after 15 years?

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.4.
1.5 APR and APY (Nominal and Effective Rates)

Learning Objectives

Here you’ll learn how to compare rates for loans and savings accounts to find more favorable deals.

In looking at an advertisement for a car you might see 2.5% APR financing on a $20,000 car. What does APR mean? What rate are they really charging you for the loan? Different banks may offer 8.1% annually, 8% compounded monthly or 7.9% compounded continuously. How much would you really be making if you put $100 in each bank? Which bank has the best deal?

Nominal and Effective Rates of Interest

A **nominal interest rate** is an interest rate in name only since a method of compounding needs to be associated with it in order to get a true effective interest rate. APR rates are nominal. **APR** stands for **Annual Percentage Rate**. The compounding periods are usually monthly, so typically \(k = 12\).

An **annual effective interest rate** is the true interest that is being charged or earned. APY rates are effective rates. **APY** stands for **Annual Percentage Yield**. It is a true rate that states exactly how much money will be earned as interest.

Banks, car dealerships and all companies will often advertise the interest rate that is most appealing to consumers who don’t know the difference between APR and APY. In places like loans where the interest rate is working against you, they advertise a nominal rate that is lower than the effective rate. On the other hand, banks want to advertise the highest rates possible on their savings accounts so that people believe they are earning more interest.

In order to calculate what you are truly being charged, or how much money an account is truly making, it is necessary to use what you have learned about compounding interest and continuous interest. Then, you can make an informed decision about what is best.

Take a credit card that advertises 19.9% APR (annual rate compounded monthly). Say you left $1000 unpaid, how much would you owe in a year?

First recognize that 19.9% APR is a nominal rate compounded monthly.

\[
FV = ? \quad PV = 1000, \quad i = .199, \quad k = 12, \quad t = 1
\]

\[
FV = 1000 \left( 1 + \frac{.199}{12} \right)^{12} \approx $1,218.19
\]

Notice that $1,218.19 is an increase of about 21.82% on the original $1,000. Many consumers expect to pay only $199 in interest because they misunderstood the term APR. The effective interest on this account is about 21.82%, which is more than advertised.

Another interesting note is that just like there are rounding conventions in this math text (4 significant digits or dollars and cents), there are legal conventions for rounding interest rate decimals. Many companies include an additional 0.0049% because it rounds down for advertising purposes, but adds additional cost when it is time to pay up. For the purposes of this concept, ignore this addition.
Examples

Example 1

Earlier, you were asked about financing a car and the difference between APR and APY. A loan that offers 2.5%
APR that compounds monthly is really charging lightly more than 2.5% of the initial loan per year.

\[
(1 + \frac{0.025}{12})^{12} \approx 1.025288
\]

They are really charging about 2.529%.

The table below shows the APY calculations for three different banks offering 8.1% annually, 8% compounded monthly and 7.9% compounded continuously.

<table>
<thead>
<tr>
<th>Bank A</th>
<th>Bank B</th>
<th>Bank C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( FV = PV (1 + i)^t )</td>
<td>( FV = PV \left( 1 + \frac{i}{k} \right)^{kt} )</td>
<td>( FV = PV \cdot e^{rt} )</td>
</tr>
<tr>
<td>( FV = 100(1 + 0.081) )</td>
<td>( FV = 100 \left( 1 + \frac{0.08}{12} \right)^{12} )</td>
<td>( FV = 100e^{0.079} )</td>
</tr>
<tr>
<td>( FV \approx 108.299 )</td>
<td>( FV \approx 108.299 )</td>
<td>( FV \approx 108.22 )</td>
</tr>
<tr>
<td>( APY = 8.1% )</td>
<td>( APY \approx 8.299% )</td>
<td>( APY \approx 8.22% )</td>
</tr>
</tbody>
</table>

Even though Bank B does not seem to offer the best interest rate, or the most advantageous compounding strategy, it still offers the highest yield to the consumer.

Example 2

Three banks offer three slightly different savings accounts. Calculate the Annual Percentage Yield for each bank and choose which bank would be best to invest in.

Bank A offers 7.1% annual interest.

Bank B offers 7.0% annual interest compounded monthly.

Bank C offers 6.98% annual interest compounded continuously.

Since no initial amount is given, choose a \( PV \) that is easy to work with like $1 or $100 and test just one year so \( t = 1 \). Once you have the future value for 1 year, you can look at the percentage increase from the present value to determine the APY.

<table>
<thead>
<tr>
<th>Bank A</th>
<th>Bank B</th>
<th>Bank C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1.5. APR and APY (Nominal and Effective Rates)

<table>
<thead>
<tr>
<th>Table 1.8: (continued)</th>
</tr>
</thead>
</table>

\[
FV = PV(1 + i)^t \\
FV = 100(1 + 0.071) \\
FV = $107.1 \\
APY = 7.1\% \\
\]

\[
FV = PV \left(1 + \frac{i}{k}\right)^{kt} \\
FV = 100 \left(1 + \frac{0.071}{12}\right)^{12} \\
FV \approx 107.229 \\
APY \approx 7.2290\% \\
\]

\[
FV = PV \cdot e^{rt} \\
FV = 100e^{0.0698} \\
FV \approx 107.2294 \\
APY \approx 7.2294\% \\
\]

Bank A compounded only once per year so the APY was exactly the starting interest rate. However, for both Bank B and Bank C, the APY was higher than the original interest rates. While the APY’s are very close, Bank C offers a slightly more favorable interest rate to an investor.

**Example 3**

The APY for two banks are the same. What nominal interest rate would a monthly compounding bank need to offer to match another bank offering 4% compounding continuously?

Solve for APY for the bank where all information is given, the continuously compounding bank.

\[
FV = PV \cdot e^{rt} = 100 \cdot e^{0.04} \approx 104.08 \\
\]

The APY is about 4.08%. Now you will set up an equation where you use the 104.08 you just calculated, but with the other banks interest rate.

\[
FV = PV \left(1 + \frac{i}{k}\right)^{kt} \\
104.08 = 100 \left(1 + \frac{i}{12}\right)^{12} \\
i = 12 \left[\left(\frac{104.08}{100}\right)^{\frac{1}{12}} - 1\right] \approx 0.0400667 \\
\]

The second bank will need to offer slightly more than 4% to match the first bank.

**Example 4**

Which bank offers the best deal to someone wishing to deposit money?

- Bank A, offering 4.5% annually compounded
- Bank B, offering 4.4% compounded quarterly
- Bank C, offering 4.3% compounding continuously

The following table shows the APY calculations for the three banks.

<table>
<thead>
<tr>
<th>Table 1.9:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank A</td>
</tr>
<tr>
<td>------------</td>
</tr>
<tr>
<td>7.1%</td>
</tr>
<tr>
<td>7.2290%</td>
</tr>
<tr>
<td>7.2294%</td>
</tr>
</tbody>
</table>
## Table 1.9: (continued)

<table>
<thead>
<tr>
<th>Formula 1</th>
<th>Formula 2</th>
<th>Formula 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( FV = PV (1 + i)^t )</td>
<td>( FV = PV \left( 1 + \frac{i}{k} \right)^{kt} )</td>
<td>( FV = PV \cdot e^{rt} )</td>
</tr>
<tr>
<td>( FV = 100(1 + 0.045) )</td>
<td>( FV = 100 \left( 1 + \frac{0.044}{4} \right)^4 )</td>
<td>( FV = 100e^{0.043} )</td>
</tr>
<tr>
<td>( APY = 4.5% )</td>
<td>( APY \approx 4.473% )</td>
<td>( APY \approx 4.394% )</td>
</tr>
</tbody>
</table>

Bank B offers the best interest rate.

### Example 5

What is the effective rate of a credit card interest charge of 34.99% APR compounded monthly? 
\[
(1 + \frac{34.99}{12})^{12} \approx 1.4118 \text{ or a 41.18\% effective interest rate.}
\]

### Review

For problems 1-4, find the APY for each of the following bank accounts.

1. Bank A, offering 3.5\% annually compounded.
2. Bank B, offering 3.4\% compounded quarterly.
3. Bank C, offering 3.3\% compounded monthly.
4. Bank D, offering 3.3\% compounding continuously.

5. What is the effective rate of a credit card interest charge of 21.99\% APR compounded monthly? 
6. What is the effective rate of a credit card interest charge of 16.89\% APR compounded monthly? 
7. What is the effective rate of a credit card interest charge of 18.49\% APR compounded monthly? 
8. The APY for two banks are the same. What nominal interest rate would a monthly compounding bank need to offer to match another bank offering 3% compounding continuously? 
9. The APY for two banks are the same. What nominal interest rate would a quarterly compounding bank need to offer to match another bank offering 1.5% compounding continuously? 
10. The APY for two banks are the same. What nominal interest rate would a daily compounding bank need to offer to match another bank offering 2% compounding monthly? 
11. Explain the difference between APR and APY. 
12. Give an example of a situation where the APY is higher than the APR. Explain why the APY is higher. 
13. Give an example of a situation where the APY is the same as the APR. Explain why the APY is the same. 
14. Give an example of a situation where you would be looking for the highest possible APY. 
15. Give an example of a situation where you would be looking for the lowest possible APY.

### Review (Answers)

To see the Review answers, open this PDF file and look for section 13.5.
Learning Objectives

Here you’ll learn how to compute future values of periodic payments.

Sally knows she can earn a nominal rate of 6% convertible monthly in a retirement account, and she decides she can afford to save $1,500 from her paycheck every month. How can you use geometric series to simplify the calculation of finding the future value of all these payments? How much money will Sally have saved in 30 years?

Annuity

An **annuity** is a series of equal payments that occur periodically. The word annuity comes from annual which means yearly. You will start by working with payments that occur once at the end of each year and then delve deeper to payments that occur monthly or any period.

Assume an investor saves $R$ dollars at the end of each year for $t$ years in an account that earns $i$ interest per period.

- The first payment $R$ will be in the bank account for $t - 1$ years and grow to be: $R(1 + i)^{t-1}$
- The second payment $R$ will be in the bank account for $t - 2$ years and grow to be: $R(1 + i)^{t-2}$
- This pattern continues until the last payment of $R$ that is deposited in the account right at $t$ years, so it doesn’t earn any interest at all.

The account balance at this point in the future (Future Value, $FV$) is the sum of each individual $FV$ of all the payments:

$$FV = R + R(1 + i)^1 + R(1 + i)^2 + \cdots + R(1 + i)^{t-2} + R(1 + i)^{t-1}$$

Recall that a geometric series with initial value $a$ and common ratio $r$ with $n$ terms has sum:

$$a + ar + ar^2 + \cdots + ar^{n-1} = a \cdot \frac{1-r^n}{1-r}$$

So, a geometric series with starting value $R$ and common ratio $(1 + i)$ has sum:

$$FV = R \cdot \frac{1 - (1 + i)^n}{1 - (1 + i)}$$

$$= R \cdot \frac{1 - (1 + i)^n}{-i}$$

$$= R \cdot \frac{(1 + i)^n - 1}{i}$$

This formula describes the relationship between $FV$ (the account balance in the future), $R$ (the annual payment), $n$ (the number of years) and $i$ (the interest per year).
The formula is extraordinarily flexible and will work even when payments occur monthly instead of yearly by rethinking what, \( R, i \) and \( n \) mean. The resulting Future Value will still be correct. If \( R \) is monthly payments, then \( i \) is the interest rate per month and \( n \) is the number of months.

Take an IRA (special type of savings account). If Lenny saves $5,000 a year at the end of each year for 35 years at an interest rate of 4%, he can determine what his Future Value will be using the formula.

\[
R = 5,000, \ i = 0.04, \ n = 35, \ FV = ?
\]

\[
FV = R \cdot \frac{(1 + i)^n - 1}{i}
\]

\[
FV = 5,000 \cdot \frac{(1 + 0.04)^{35} - 1}{0.04}
\]

\[
FV = $368,281.12
\]

Examples

Example 1

Earlier, you were given a problem where Sally wanted to know how much she will have if she can earn a nominal 6% interest rate compounded monthly in a retirement account where she decides to save $1500 from her paycheck every month for thirty years.

\[
FV = ?, \ i = \frac{0.06}{12} = 0.005, \ n = 30 \cdot 12 = 360, \ R = 1,500
\]

\[
FV = R \cdot \frac{(1 + i)^n - 1}{i}
\]

\[
FV = 1,500 \cdot \frac{(1 + 0.005)^{360} - 1}{0.005}
\]

\[
FV \approx 1,506,772.56
\]
Example 2

How long does Mariah need to save if she wants to retire with a million dollars and saves $10,000 a year at 5% interest?

\[ FV = 1,000,000, \ R = 10,000, \ i = 0.05, \ n = ? \]

\[
FV = R \cdot \frac{(1+i)^n - 1}{i}
\]

\[
1,000,000 = 10,000 \cdot \frac{(1+0.05)^n - 1}{0.05}
\]

\[
100 = \frac{(1+0.05)^n - 1}{0.05}
\]

\[
5 = (1+0.05)^n - 1
\]

\[
6 = (1+0.05)^n
\]

\[
n = \frac{\ln 6}{\ln 1.05} \approx 36.7 \text{ years}
\]

Example 3

How much will Peter need to save each month if he wants to buy an $8,000 car with cash in 5 years? He can earn a nominal interest rate of 12% compounded monthly.

In this situation you will do all calculations in months instead of years. An adjustment in the interest rate and the time is required and the answer needs to be clearly interpreted at the end.

\[ FV = 8,000, \ R = ?, \ i = \frac{0.12}{12} = 0.01, \ n = 5 \cdot 12 = 60 \]

\[
FV = R \cdot \frac{(1+i)^n - 1}{i}
\]

\[
8,000 = R \cdot \frac{(1+0.01)^{60} - 1}{0.01}
\]

\[
R = \frac{8,000 \cdot 0.01}{(1+0.01)^{60} - 1} \approx 97.96
\]

Peter will need to save about $97.96 every month.

Example 4

At the end of each quarter, Fermin makes a $200 deposit into a mutual fund. If his investment earns 8.1% interest compounded quarterly, what will his annuity be worth in 15 years?

Quarterly means 4 times per year.

\[ FV =?, \ R = 200, \ i = \frac{0.081}{4}, \ n = 60 \]

\[
FV = 200 \cdot \frac{(1+\frac{0.081}{4})^{60} - 1}{\frac{0.081}{4}} \approx $23,008.71
\]
Example 5

What interest rate compounded semi-annually is required to grow $25 semi-annual payments to $500 in 8 years?

\[ FV = 500, \quad R = 25, \quad i = ?, \quad n = 8 \cdot 2 = 16. \]

Note that the calculation will be done in months. At the end you will convert your answer to years.

\[
FV = R \cdot \frac{(1+i)^n - 1}{i}
\]

\[
500 = 25 \cdot \frac{(1+i)^{16} - 1}{i}
\]

\[
20i = (1+i)^{16} - 1
\]

\[
0 = (1+i)^{16} - 20i - 1
\]

Using a graphing calculator, we find this equation has roots at \( i = 0 \) and \( i = 0.0290 \). Since \( i \neq 0 \), the semi-annual interest rate is \( i = 0.0290 = 2.90\% \) for a nominal annual interest rate of 5.80%.

Review

1. At the end of each month, Rose makes a $400 deposit into a mutual fund. If her investment earns 6.1% interest compounded monthly, what will her annuity be worth in 30 years?

2. What interest rate compounded quarterly is required to grow a $40 quarterly payment to $1000 in 5 years?

3. How many years will it take to save $10,000 if Sal saves $50 every month at a 2% monthly interest rate?

4. How much will Bob need to save each month if he wants to buy a $33,000 car with cash in 5 years? He can earn a nominal interest rate of 12% compounded monthly.

5. What will the future value of his IRA be if Cal saves $5,000 a year at the end of each year for 35 years at an interest rate of 8%?

6. How long does Kathy need to save if she wants to retire with four million dollars and saves $10,000 a year at 8% interest?

7. What interest rate compounded monthly is required to grow a $416 monthly payment to $80,000 in 10 years?

8. Every six months, Shanice makes a $1000 deposit into a mutual fund. If her investment earns 5% interest compounded semi-annually, what will her annuity be worth in 25 years?

9. How much will Jen need to save each month if she wants to put $60,000 down on a house in 5 years? She can earn a nominal interest rate of 8% compounded monthly.

10. How long does Adrian need to save if she wants to retire with three million dollars and saves $5,000 a year at 10% interest?

11. What will the future value of her IRA be if Vanessa saves $3,000 a year at the end of each year for 40 years at an interest rate of 6.7%?

12. At the end of each quarter, Justin makes a $1,500 deposit into a mutual fund. If his investment earns 4.5% interest compounded quarterly, what will her annuity be worth in 35 years?

13. What will the future value of his IRA be if Ted saves $3,500 a year at the end of each year for 25 years at an interest rate of 5.8%?

14. What interest rate compounded monthly is required to grow a $300 monthly payment to $1,000,000 in 35 years?

15. How much will Katie need to save each month if she wants to put $55,000 down in cash on a house in 2 years? She can earn a nominal interest rate of 6% compounded monthly.
1.6. Annuities

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.6.
Learning Objectives

Here you’ll learn how to compute present values of equal periodic payments.

Many people buy houses they cannot afford. This causes major problems for both the banks and the people who have their homes taken. In order to make wise choices when you buy a house, it is important to know how much you can afford to pay each period and calculate a maximum loan amount.

Joanna knows she can afford to pay $12,000 a year for a house loan. Interest rates are 4.2% annually and most house loans go for 30 years. What is the maximum loan she can afford? What will she end up paying after 30 years?

Annuities for Loans

The present value can be found from the future value using the regular compound growth formula:

\[ PV(1 + i)^n = FV \]
\[ PV = \frac{FV}{(1 + i)^n} \]

You also know the future value of an annuity:

\[ FV = R \cdot \frac{(1+i)^n-1}{i} \]

So by substitution, the formula for the present value of an annuity is:

\[ PV = R \cdot \frac{(1+i)^n-1}{i} \cdot \frac{1}{(1+i)^n} = R \cdot \frac{(1+i)^n-1}{i(1+i)^n} = R \cdot \frac{1-(1+i)^{-n}}{i} \]

The present value of a series of equal payments \( R \) with interest rate \( i \) per period for \( n \) periods is:

\[ PV = R \cdot \frac{1-(1+i)^{-n}}{i} \]

This formula can also be used to find out other information such as how much a regular payment should be and how long it will take to pay off a loan.

Take a $1,000,000 house loan over 30 years with a nominal interest rate of 6% compounded monthly. You are not given the monthly payments, \( R \). To find \( R \), solve for \( R \) in the formula given above.

\[ PV = 1,000,000, \ R = ?, \ i = 0.005, \ n = 360 \]

\[ PV = R \cdot \frac{1-(1+i)^{-n}}{i} \]

\[ 1,000,000 = R \cdot \frac{1-(1+0.005)^{-360}}{0.005} \]

\[ R = \frac{1,000,000 \cdot 0.005}{1-(1+0.005)^{-360}} \approx 5995.51 \]
It is remarkable that in order to pay off a $1,000,000 loan you will have to pay $5,995.51 a month, every month, for thirty years. After 30 years, you will have made 360 payments of $5995.51, and therefore will have paid the bank more than $2.1 million, more than twice the original loan amount. It is no wonder that people can get into trouble taking on more debt than they can afford.

Examples

Example 1

Earlier, you were asked about how much Joanna can afford to take out in a loan. Joanna knows she can afford to pay $12,000 a year to pay for a house loan. Interest rates are 4.2% annually and most house loans go for 30 years. What is the maximum loan she can afford? What does she end up paying after 30 years? You can use the present value formula to calculate the maximum loan:

\[ PV = 12,000 \cdot \frac{1 - (1 + 0.042)^{-30}}{0.042} \approx 202,556.98 \]

For 30 years she will pay $12,000 a year. At the end of the 30 years she will have paid $12,000 \cdot 30 = $360,000 total

Example 2

How long will it take to pay off a $20,000 car loan with a 6% annual interest rate compounded monthly if you pay it off in monthly installments of $500? What about if you tried to pay it off in monthly installments of $100?

\[ PV = 20,000, \ R = 500, \ i = \frac{0.06}{12} = 0.005, \ n =? \]
\[ PV = R \cdot \frac{1 - (1 + i)^{-n}}{i} \]

\[
\begin{align*}
20,000 &= 500 \cdot \frac{1 - (1 + 0.005)^{-n}}{0.005} \\
0.2 &= 1 - (1 + 0.005)^{-n} \\
(1 + 0.005)^{-n} &= 0.8 \\
R &= \frac{-\ln(0.8)}{-\ln(1.005)} \approx 44.74 \text{ months}
\end{align*}
\]

For the $100 case, if you try to set up an equation and solve, there will be an error. This is because the interest on $20,000 is exactly $100 and so every month the payment will go to only paying off the interest. If someone tries to pay off less than $100, then the debt will grow.

**Example 3**

It saves money to pay off debt faster in order to save money on interest. As shown earlier, interest can more than double the cost of a 30 year mortgage. This example shows how much money can be saved by paying off more than the minimum.

Suppose a $300,000 loan has 6% interest convertible monthly with monthly payments over 30 years. What are the monthly payments? How much time and money would be saved if the monthly payments were larger by a factor of \( \frac{13}{12} \)? This is like making 13 payments a year instead of just 12. First you will calculate the monthly payments if 12 payments a year are made.

\[
\begin{align*}
PV &= R \cdot \frac{1 - (1 + i)^{-n}}{i} \\
300,000 &= R \cdot \frac{1 - (1 + 0.005)^{-360}}{0.005} \\
R &= \frac{1,798.65}{1948.54} = 0.9287 \\
&= \frac{-\ln(0.8)}{-\ln(1.005)} \approx 44.74 \text{ months}
\end{align*}
\]

After 30 years, you will have paid $647,514.57, more than twice the original loan amount.

If instead the monthly payment was \( \frac{13}{12} \cdot 1,798.65 = 1948.54 \), you would pay off the loan faster. In order to find out how much faster, you will make your unknown.

\[
\begin{align*}
PV &= R \cdot \frac{1 - (1 + i)^{-n}}{i} \\
300,000 &= 1948.54 \cdot \frac{1 - (1 + 0.005)^{-n}}{0.005} \\
0.7698 &= 1 - (1 + 0.005)^{-n} \\
(1 + 0.005)^{-n} &= 0.23019 \\
R &= \frac{-\ln(0.23019)}{-\ln(1.005)} \approx 294.5 \text{ months}
\end{align*}
\]

294.5 months is about 24.5 years. Paying fractionally more each month saved more than 5 years of payments.

\[
294.5 \text{ months} \cdot \frac{1,948.54}{1,798.65} = $573,847.99
\]

The loan ends up costing $573,847.99, which saves you more than $73,000 over the total cost if you had paid over 30 years.
1.7. Annuities for Loans

**Example 4**

Mackenzie obtains a 15 year student loan for $160,000 with 6.8% interest. What will her yearly payments be?

\[ PV = 160,000, \quad R = ?, \quad n = 15, \quad i = 0.068 \]

\[
160,000 = R \cdot \frac{1 - (1 + 0.068)^{-15}}{0.068}
\]

\[ R \approx 17,345.88 \]

**Example 5**

How long will it take Francisco to pay off a $16,000 credit card bill with 19.9% APR if he pays $800 per month?

Note: APR in this case means nominal rate convertible monthly.

\[ PV = 16,000, \quad R = 800, \quad n = ?, \quad i = \frac{0.199}{12} \]

\[
16,000 = 800 \cdot \frac{1 - (1 + \frac{0.199}{12})^{-n}}{\frac{0.199}{12}}
\]

\[ n = 24.50 \text{ months} \]

**Review**

For problems 1-10, find the missing value in each row using the present value for annuities formula.

**Table 1.10:**

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>PV</th>
<th>R</th>
<th>n (years)</th>
<th>i (annual)</th>
<th>Periods per year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$4,000</td>
<td></td>
<td>7</td>
<td>1.5%</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>$15,575</td>
<td></td>
<td>5</td>
<td>5%</td>
<td>4</td>
</tr>
<tr>
<td>3.</td>
<td>$4,500</td>
<td>$300</td>
<td>12</td>
<td>3%</td>
<td>12</td>
</tr>
<tr>
<td>4.</td>
<td>$1,000</td>
<td></td>
<td>10</td>
<td>2%</td>
<td>1</td>
</tr>
<tr>
<td>5.</td>
<td>$16,670</td>
<td></td>
<td>10</td>
<td>10%</td>
<td>4</td>
</tr>
<tr>
<td>6.</td>
<td>$400</td>
<td></td>
<td>4</td>
<td>2%</td>
<td>12</td>
</tr>
<tr>
<td>7.</td>
<td>$315,000</td>
<td>$1,800</td>
<td></td>
<td>5%</td>
<td>12</td>
</tr>
<tr>
<td>8.</td>
<td>$500</td>
<td></td>
<td>30</td>
<td>8%</td>
<td>12</td>
</tr>
<tr>
<td>9.</td>
<td>$1,000</td>
<td></td>
<td>40</td>
<td>6%</td>
<td>4</td>
</tr>
<tr>
<td>10.</td>
<td>$10,000</td>
<td></td>
<td>6</td>
<td>7%</td>
<td>12</td>
</tr>
</tbody>
</table>

11. Charese obtains a 15 year student loan for $200,000 with 6.8% interest. What will her yearly payments be?

12. How long will it take Tyler to pay off a $5,000 credit card bill with 21.9% APR if he pays $300 per month?

Note: APR in this case means nominal rate convertible monthly.

13. What will the monthly payments be on a credit card debt of $5,000 with 24.99% APR if it is paid off over 3 years?

14. What is the monthly payment of a $300,000 house loan over 30 years with a nominal interest rate of 2% convertible monthly?
15. What is the monthly payment of a $270,000 house loan over 30 years with a nominal interest rate of 3% convertible monthly?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 13.7.

The effects of interest on lump sum deposits and periodic deposits were explored. The key idea was that a dollar today is worth more than a dollar in a year.